Nourdin-Peccati analysis on Wiener and Wiener-Poisson space for general distributions

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Abstract

Given a reference random variable, we study the solution of its Stein equation and obtain universal bounds on its first and second derivatives. We then extend the analysis of Nourdin and Peccati by bounding the Fortet-Mourier and Wasserstein distances from more general random variables such as members of the Exponential and Pearson families. Using these results, we obtain non-central limit theorems, generalizing the ideas applied to their analysis of convergence to Normal random variables. We do these in both Wiener space and the more general Wiener-Poisson space. In the former, we study conditions for convergence under several particular cases and characterize when two random variables have the same distribution. As an example, we apply this tool to bilinear functionals of Gaussian subordinated fields where the underlying process is a fractional Brownian motion with Hurst parameter bigger than 1/2. In the latter space we give sufficient conditions for a sequence of multiple (Wiener-Poisson) integrals to converge to a Normal random variable.

1 Introduction

Recent years have seen exciting research on combining Stein's method with Malliavin calculus in proving central and non-central limit theorems. The delicate combination of these tools can be attributed to Nourdin and Peccati who intertwined an integration by parts formula from Malliavin calculus with an ordinary differential equation called a Stein equation. Much work has been done to compare Normal or Gamma random variables (r.v.'s) with another r.v. (having unknown distribution). See [13], [16], [20], [21] for results on the convergence of multiple (Wiener) integrals to a standard Normal or Gamma law. [3] and [27] discuss Cramer's theorem for Normal and Gamma distributions applied to multiple integrals. [29] gives probability tail bounds in terms of the Normal probability tail, with [8] applying the same techniques to give tail bounds in terms of the probability tail of other r.v.'s (e.g. Pearson distributions).

In [14], Nourdin and Peccati found a clever link between Stein's method and Malliavin calculus. This was used to derive the Nourdin-Peccati upper bound (NP bound) on the Wasserstein, Total Variation, Fortet-Mourier and Kolmogorov distances of a generic r.v. from a Normal r.v., and lay the groundwork for comparisons to a more general r.v. (with such results leading to non-central limit theorems). These authors and Reinert (see [17]) applied this NP bound to obtain a second order Poincaré-type inequality useful in proving central limit theorems (CLTs) in Wiener space. Specifically, they proved CLTs for linear functionals of Gaussian subordinated fields. Particular instances are when the subordinated process is fractional Brownian motion (fBm) or the solution to the Ornstein-Uhlenbeck (O-U) SDE driven by fBm. They also characterized convergence in distribution to a Normal r.v. for multiple stochastic integrals.

Later in [22] these ideas were applied to prove the NP bound in Poisson space (pure jump processes), which was used to obtain Berry-Esséen bounds for arbitrary tensor powers of O-U kernels. Keeping in line with attempts to extend these results as far as possible, [30] proved an NP bound in Wiener-Poisson space. The author applied similar ideas found in [17] to derive a second order Poincaré-type inequality and use it to prove CLTs for a continuous average of a product of two O-U processes (one in Wiener space and the other in Poisson space) which lives in the second chaos of Wiener-Poisson space. Also, it was proved that under mild conditions, the small jumps part of a functional in the first Poisson chaos is approximately equal in law to a functional in the first Wiener chaos with the same kernel (useful when simulating a fractional Lévy process as a process with finitely many jumps plus a fBm). All these results show the importance of this NP bound and the potential it has as an effective tool in proving non-central limit theorems, CLTs and characterizations.

Let Z be absolutely continuous with respect to Lebesgue measure. For our purposes, we can think of Z as a "well-behaved" r.v. (e.g. it lives in Wiener space and we know its density). Typical instances are when Z is Normal, Gamma, or another member of the Pearson family of distributions. X is another r.v. whose properties are not as easy to determine as with Z, our "target" r.v.. We may have a hunch that X has the same distribution as Z, or in the case of sequences, a belief that $\{X_n\}$ converges in law to the distribution of Z. We thus want to compare X with Z. How different are the laws of X and Z for instance (and we need to make precise the sense in which they are different)? What conditions will ensure that X has the same law as Z? For a sequence $\{X_n\}$, what sufficient conditions ensure convergence to Z in distribution? In this regard, we wish to measure the distance between (the laws of) X and Z by a metric $d_{\mathcal{H}}$ which induces a topology that is equal to or stronger than the topology of convergence in distribution: if $d_{\mathcal{H}}(X_n, Z) \to 0$, then $X_n \to Z$ in distribution.

The motivation for this paper is to find the widest generalization of the NP bound by applying it to a target r.v. which is neither Normal nor Gamma, and in both Wiener space and Wiener-Poisson space. This is worked out in [10] but the conditions needed to apply the NP bound are quite restrictive (it was also carried out only in Wiener space). The conditions we are introducing here are more general, and are still wide enough in scope to cover a Z belonging to the Exponential family or the Pearson family. We point out that Wiener-Poisson space is more inclusive than Wiener space (which can be identified with a subspace of the former). In fact, it includes processes with jumps, and therefore considers Poisson space too as a subspace (also by identification). Nevertheless, even if Wiener space is less general, we can apply our techniques to a wider class of target r.v. Z than in Wiener-Poisson space (which requires boundedness of the second derivative of the solution of the Stein's equation, something not needed in Wiener space).

Our main results are the NP bounds on $d_{\mathcal{H}}(X, Z)$ in Wiener space and in Wiener-Poisson space. The main result in Wiener space (Theorem 12) is

$$\begin{aligned} d_{\mathcal{H}}(X, Z) &\leq k\mathbb{E}\left|g_*(X) - g_X\right| \\ &\leq k\sqrt{\left|\mathbb{E}\left[g_*(X)^2\right] - \mathbb{E}\left[g_*(Z)^2\right]\right| + \left|\mathbb{E}\left[XG_*(X)\right] - \mathbb{E}\left[ZG_*(Z)\right]\right| + \left|\mathbb{E}\left[g_X^2\right] - \mathbb{E}\left[g_Z^2\right]\right|}. \end{aligned}$$

The main result in Wiener-Poisson space (Theorem 26) is

$$d_{\mathcal{H}}(X, Z) \leq k\left(\mathbb{E}\left|g_*(X) - g_X\right| + \mathbb{E}\left[\left\langle\left|x\,(DX)^2\right|, \left|-DL^{-1}X\right|\right\rangle_{\mathfrak{H}}\right]\right).$$

Here, $g_X := \mathbb{E}[\langle DX, -DL^{-1}X\rangle_{\mathfrak{H}}|X]$ and $g_Z := \mathbb{E}[\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}}|Z]$ are random variables defined using Malliavin calculus operators, specifically, the Malliavin derivative D and the inverse of the infinitesimal generator L of the O-U semigroup. The definitions parallel each other, but it would be helpful to think of g_X

as an object belonging exclusively to X, and g_Z to Z. On the other hand, $g_*(\cdot) := \mathbb{E}[\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} | Z = \cdot]$ is a function whose support is the support of Z, taking on nonnegative numbers as values. It will depend only on the density of Z, and is independent of the structure of X. As such, it is an object belonging solely to Z. In the second term of the first bound above, G_* is an antiderivative of g_*, provided it exists. We can make sense of the NP bounds in the following way: if we want to know how different the laws of X and Z are, then we need to know how different (in the L^1 sense) $g_X = \mathbb{E}[\langle DX, -DL^{-1}X\rangle_{\mathfrak{H}} | X]$ and $g_*(X) = \mathbb{E}[\langle DZ, -DL^{-1}Z\rangle_{\mathfrak{H}} | Z = X]$ are. In Wiener-Poisson space, we consider in addition how close the jump part $\mathbb{E}[\langle |x\,(DX)^2|, |-DL^{-1}X|\rangle_{\mathfrak{H}}]$ is to 0, which makes sense since Z belongs to Wiener space (subspace of the Wiener-Poisson space without jumps).

In our bounds above, k is a constant that does not depend on X but on Z and the metric we are using. For convergence problems, we do not need its specific value since the convergence will follow from the convergence of $\mathbb{E}\,|g_*(X_n) - g_{X_n}|$ to 0. This presupposes we have such a constant k. This constant appears as a bound ($\|\phi\|_\infty \leq k$) for some function ϕ, which is related to the solution of the underlying Stein equation. In particular, since we have a Stein equation for each Z (the target r.v.), k depends on Z. Finding such a bound k is easy when Z is Normal: g_* is constant, and consequently, the Stein equation is simpler. If g_* vanishes at a finite endpoint of the support of Z, the challenge now is to find a bound for $\|\phi/g_*\|_\infty$. To the best of our knowledge, [10] (Kusuoka and Tudor) presents the first attempt to find such a sup norm bound when Z is not Normal. Their result is presented below as Lemma 7. In Theorem 9 we improve their result, and this paves the way for the needed bound we stated for the general non-Normal case.

The paper is organized as follows. In Section 2, we review the operators we need from Malliavin calculus. We also define the functions g_* and G_* as well as the random variables g_X and g_Z, studying carefully their properties (needed in the subsequent sections). Section 3 contains preliminaries on Stein's method. Here we find universal bounds on the first and second derivatives of the solution of the general Stein equation. Our main result in Wiener space is in Section 4, where we give a tractable upper inequality which is easier to compute. We also characterize when the law of X is the same as that of Z. Said result is applied to specific cases when g_* is a polynomial and when $\{X_n\}$ is a sequence of multiple integrals. As an example, we prove the convergence of a bilinear functional of a Gaussian subordinated field to a χ^2 r.v. by computing some moments and showing their convergence to desired values. In Section 5, we extend the main result to the more general Wiener-Poisson space. Here, we work out some sufficient conditions for convergence to a Normal law and convergence of the fourth moment.

2 Elements of Malliavin calculus and tools

For the sake of completeness, we include here a brief survey of the needed Malliavin calculus objects. The r.v. $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$, to be constructed for $F = Z$ and for $F = X$, is a key element that bridges Stein's method and Malliavin calculus. D is the Malliavin derivative operator and L is the generator of the Ornstein-Uhlenbeck semigroup.

2.1 Wiener space

Nualart presents in Chapter 1 of [19] a very good exposition on Malliavin calculus in Wiener space. We mention here the elements that we need. Let $\mathfrak{H}$ be a real separable Hilbert space. Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which $W = \{W(h) : h \in \mathfrak{H}\}$ is an isonormal Gaussian process. By definition, this means W is a centered Gaussian family such that $\mathbb{E}[W(h_1)\,W(h_2)] = \langle h_1, h_2\rangle_{\mathfrak{H}}$. We may also assume that $\mathcal{F}$ is the σ-field generated by W. The white noise case is when $\mathfrak{H} = L^2(T, \mathcal{B}, \mu)$ where $(T, \mathcal{B})$ is a measurable space and μ is a σ-finite atomless measure. The Gaussian process W is then characterized by the family of r.v.'s

$\{W(A) : A \in \mathcal{B}[0,T], \mu(A) < \infty\}$ where $W(A) = W(\mathbf{1}_A)$; we write $W_t = W\left(\mathbf{1}_{[0,t]}\right)$. We can then think of W as an $L^2(\Omega, \mathcal{F}, \mathbb{P})$ random measure on $(T, \mathcal{B})$. This is called the white noise measure based on μ.

The q^{th} Hermite polynomial H_q is given by $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}\left(e^{-x^2/2}\right)$ for $q \geq 1$ and $H_0(x) = 1$. The q^{th} Wiener chaos $\mathcal{H}_q$ is defined as the subspace of $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the r.v.'s $\left\{H_q(W(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\right\}$. In the white noise case $\mathfrak{H} = L^2_\mu([0,1])$, each Wiener chaos consists of iterated multiple (Wiener) integrals

$$I_q(f) := n! \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{q-1}} f(t_1, t_2, \ldots, t_q)\, dW_{t_q} \cdots dW_{t_2} dW_{t_1}$$

with respect to W, where $f \in \mathfrak{H}^{\odot q}$ is a symmetric nonrandom kernel. When f is nonsymmetric, we let $\widetilde{f}$ denote its symmetrization, and $I_q(f) = I_q(\widetilde{f})$.

All elements of $\mathcal{H}_1$ are Gaussian and all elements of $\mathcal{H}_0$ are deterministic. It is well-known that $L^2(\Omega)$ can be decomposed into an infinite orthogonal sum of the Wiener chaoses, i.e. $L^2(\Omega) = \oplus_{q=0}^{\infty} \mathcal{H}_q$. In the white noise case, any $F \in L^2(\Omega)$ admits a Wiener chaos decomposition of multiple integrals

$$F = \sum_{q=0}^{\infty} I_q(f_q) \tag{1}$$

where each symmetric $f_q \in \mathfrak{H}^{\odot q} = L^2_\mu(T^q)$ is uniquely determined by F. Note that $I_0(f_0) = f_0 = \mathbb{E}[F]$ and $\mathbb{E}[I_q(f_q)] = 0$ for $q \geq 1$.

Consider an orthonormal system $\{e_k : k \geq 1\}$ in $\mathfrak{H}$. For $f \in \mathfrak{H}^{\otimes p}$ and $g \in \mathfrak{H}^{\otimes q}$, the contraction of order $r \leq \min\{p, q\}$ is the element $f \otimes_r g \in \mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1, \ldots, i_r}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

Even if f and g are symmetric, $f \otimes_r g$ may be nonsymmetric so we denote its symmetrization by $f \widetilde{\otimes}_r g$. In the white noise case $\mathfrak{H} = L^2_\mu(T)$, the contraction is given by integrating out r variables. Thus, if $f \in L^2_\mu(T^p)$ and $q \in L^2_\mu(T^q)$, we have $f \otimes_r g \in L^2_\mu\left(T^{p+q-2r}\right)$ and

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{T^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r)\, g(t_{p+1}, \ldots, t_{p+q-r}, s_1, \ldots, s_r)\, d\mu(s_1) \cdots d\mu(s_r).$$

The product of two multiple integrals is

$$I_q(f) I_p(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{q+p-2r}(f \otimes_r g). \tag{2}$$

The Malliavin derivative of a random variable $F \in L^2(\Omega)$ is an $\mathfrak{H}$-valued random variable denoted by DF. In the white noise case $\mathfrak{H} = L^2_\mu(T)$, if $F = I_1(f) = \int_T f(t)\, dW_t$, then D maps F to an $L^2_\mu(T)$-valued element: $D_r F = f(r)$ for $r \in T$. In general, if $F \in L^2(\Omega)$ admits the decomposition (1), then

$$D_r F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(r, \cdot)). \tag{3}$$

We denote by $\mathbb{D}^{1,2}$ the domain of D in $L^2(\Omega)$. F with the above decomposition is in $\mathbb{D}^{1,2}$ if and only if $\mathbb{E}\left[\|DF\|^2_{L^2_\mu(T)}\right] = \sum_{q=1}^{\infty} q \cdot q! \|f_q\|^2_{L^2_\mu(T^q)} < \infty$. D satisfies the chain rule formula: $D(f(F)) = f'(F)\, DF$

when $F \in \mathbb{D}^{1,2}$ and f is differentiable with bounded derivative. One may relax this to almost everywhere differentiability of f as long as F has an absolutely continuous law.

D has an adjoint, the divergence operator δ, so that if $F \in \operatorname{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$, then $\delta(F) \in L^2(\Omega)$ and $\mathbb{E}[\delta(F)G] = \mathbb{E}\left[\langle F, DG\rangle_{\mathfrak{H}}\right]$ for any $G \in \mathbb{D}^{1,2}$. In the white noise case, δ is called the Skorohod integral: for $F \in \operatorname{Dom}\delta \subset L^2_{\mu\times\mathbb{P}}(T \times \Omega)$ with chaos representation $F(t) = \sum_{q=0}^{\infty} I_q(f_q(t,\cdot))$ where each $f_q \in L^2_{\mu^{\otimes(q+1)}}$ is symmetric in the last q variables, $\delta(F) = \sum_{q=0}^{\infty} I_{q+1}\left(\widetilde{f_q}\right)$ if $\sum_{q=0}^{\infty}(q+1)!\left\|\widetilde{f_q}\right\|^2_{L^2_{\mu^{\otimes(q+1)}}} < \infty$, i.e. $F \in \operatorname{Dom}\delta$.

One other operator we need, L, acts on F as in (1) in this way: $LF = -\sum_{q=1}^{\infty} qI_q(f_q)$. Its domain consists of F for which $\sum_{q=1}^{\infty} q^2 \cdot q!\,\|f_q\|^2_{L^2_\mu(T^q)} < \infty$. L also happens to be the infinitesimal generator of the Ornstein-Uhlenbeck semigroup T_t, defined by $T_tF = \sum_{q=0}^{\infty} e^{-qt} I_q(f_q)$. One important relation is $\delta DF = -LF$. More than L, we need its pseudo-inverse L^{-1} defined by $L^{-1}F = -\sum_{q=1}^{\infty} \frac{1}{q} I_q(f_q)$. It easily follows that $L^{-1}LF = F - \mathbb{E}[F]$.

2.2 Wiener-Poisson space

Assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which $\mathcal{L} = \{\mathcal{L}_t\}_{t\geq 0}$ is a Lévy process. By definition, this means $\mathcal{L}$ has stationary and independent increments, is continuous in probability, and $\mathcal{L}_0 = 0$. Suppose $\mathcal{L}$ is cadlag, centered, and $\mathbb{E}\left[\mathcal{L}_1^2\right] < \infty$. We may also assume $\mathcal{F}$ is generated by $\mathcal{L}$. Let $\mathcal{L}$ have Lévy triplet $\left(0, \sigma^2, \nu\right)$ and thus, Lévy-Itô decomposition $\mathcal{L}_t = \sigma W_t + \int\int_{[0,t)\times\mathbb{R}_0} x d\widetilde{N}(s,x)$ where $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion, $\widetilde{N}$ is the compensated jump measure (defined in terms of ν) and $\mathbb{R}_0 = \mathbb{R} - \{0\}$. See [1] and [23] for more about Lévy processes.

Consider now the measure μ on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ where $\mathbb{R}^+ = \{t : t \geq 0\}$ and

$$d\mu(t,x) = \sigma^2 dt \delta_0(x) + x^2 dt d\nu(x)(1 - \delta_0(x)).$$

Analogous to a Gaussian process W being extended to a random measure (which we also denoted by W) in Wiener space, $\mathcal{L}$ can be extended to a random measure M (see [9]) on $(\mathbb{R}^+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}))$. This is used to construct (in an analogous way to the Itô integral construction) an integral on step functions, and then by linearity and continuity, extended to $L^2_{\mu^{\otimes q}} = L^2\left((\mathbb{R}^+ \times \mathbb{R})^q, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R})^q, \mu^{\otimes q}\right)$. We also denote it by I_q. As in Wiener space,

1. $I_q(f) = I_q\left(\widetilde{f}\right)$;
2. I_q is linear;
3. $\mathbb{E}[I_q(f) I_p(g)] = \mathbf{1}_{\{q=p\}} q! \int_{(\mathbb{R}^+\times\mathbb{R})^q} \widetilde{f}\widetilde{g} d\mu^{\otimes q}$.

Thus, when $F = I_q(f)$, $\mathbb{E}[F^2] = \mathbb{E}[I_q(f)^2] = q!||\widetilde{f}||^2_{L_\mu^{\otimes q}}$.

Contractions are defined slightly differently. Suppose $f \in L^2_{\mu^{\otimes q}}$ and $g \in L^2_{\mu^{\otimes p}}$. Let $r \leq \min\{q,p\}$ and $s \leq \min\{q,p\} - r$. The contraction $f \otimes_r^s g \in L^2_{\mu^{\otimes(q+p-2r-s)}}$ is defined by integrating out r variables and sharing s of the remaining variables:

$$(f \otimes_r^s g)(z,u,v) = \left(\prod_{i=1}^{s} x_i\right) \langle f(\cdot, z, u), g(\cdot, z, v)\rangle_{L^2_{\mu^{\otimes r}}}$$

where $z \in (\mathbb{R}^+ \times \mathbb{R}_0)^s$, $z_i = (t_i, x_i)$, $u \in (\mathbb{R}^+ \times \mathbb{R}_0)^{q-r-s}$ and $v \in (\mathbb{R}^+ \times \mathbb{R}_0)^{p-r-s}$. Its symmetrization is $f \widetilde{\otimes}_r^s g$. We need the following product formula later (see [11] for the proof):

$$I_q(f) I_p(g) = \sum_{r=0}^{p \wedge q} \sum_{s=0}^{p \wedge q - r} r!s! \binom{p}{r}\binom{q}{r}\binom{p-r}{s}\binom{q-r}{s} I_{q+p-2r-s}\left(f \otimes_r^s g\right). \tag{4}$$

We may think of this as a more general version of the product formula (2) where we only consider $s = 0$ since there are no jump components to be shared (which appear in the definition of $f \otimes_r^s g$).

We have briefly narrated a setup parallel to what was done in Wiener space. See [24] for a more detailed exposition. This time though, we have only considered $\mathfrak{H} = L^2(\mathbb{R}^+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}), \mu)$ as underlying Hilbert space, with inner product $\langle f, g \rangle_{\mathfrak{H}} = \int_{\mathbb{R}^+ \times \mathbb{R}} f(z) g(z) \, d\mu(z)$. There is as yet no Malliavin calculus theory developed for a more general abstract Hilbert space. While we don't have a chaos decomposition via orthogonal polynomials (like Hermite polynomials in Wiener space; see [7]), we still have a comparable decomposition proved by Itô (Theorem 2, [9]): for $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$,

$$F = \sum_{q=0}^{\infty} I_q(f_q) \text{ where } f_q \in L^2_{\mu^{\otimes q}}. \tag{5}$$

With this decomposition, we can define the Malliavin derivative operator and Skorohod integral operator. Define $\operatorname{Dom} D$ as the set of $F \in L^2(\Omega)$ for which $\sum_{q=1}^{\infty} qq! \left\| f_q \right\|^2_{L^2_{\mu^{\otimes q}}} < \infty$ and

$$D_z F = \sum_{q=1}^{\infty} q I_{q-1}\left(f_q(z, \cdot)\right).$$

It is instructive to consider the derivatives $D_{t,0}$ and D_z where $z = (t, x)$ has $x \neq 0$. This will enable us to better understand the similarities, and where they end, between the Malliavin calculus of Wiener space and that of Wiener-Poisson space. See [24] and [25] for more details on the following discussion. We consider two spaces on which we can embed $\operatorname{Dom} D$. For $F \in L^2(\Omega)$, we say $F \in \operatorname{Dom} D^0$ iff $\sum_{q=1}^{\infty} qq! \int_{\mathbb{R}^+} \left\| f_q((t,0), \cdot) \right\|^2_{L^2_{\mu^{\otimes(q-1)}}} dt < \infty$ and $F \in \operatorname{Dom} D^J$ iff $\sum_{q=1}^{\infty} qq! \int_{\mathbb{R}^+ \times \mathbb{R}_0} \left\| f_q(z, \cdot) \right\|^2_{L^2_{\mu^{\otimes(q-1)}}} d\mu(z) < \infty$. In fact, $\operatorname{Dom} D = \operatorname{Dom} D^0 \cap \operatorname{Dom} D^J$. Since W and $\widetilde{N}$ are independent, we can think of Ω as a cross product of the form $\Omega_W \times \Omega_J$ where $\Omega_W = \mathcal{C}(\mathbb{R}^+)$ and Ω_J consists of the sequences $((t_1, x_1), (t_2, x_2), \ldots) \in (\mathbb{R}^+ \times \mathbb{R}_0)^{\mathbb{N}}$ (with a few other technical conditions).

- The derivative $D_{t,0}$ can be interpreted as the derivative with respect to the Brownian motion part. In fact, if $\nu = 0$, then $D_{t,0} F = \frac{1}{\sigma} D_t^W F$ where D^W is the classical Malliavin derivative (defined in Wiener space); the $\frac{1}{\sigma}$ comes from the fact that we are differentiating with respect to σW_t and not just W_t. From the isometry $L^2(\Omega) \simeq L^2\left(\Omega_W; L^2(\Omega_J)\right)$, consider $F \in L^2(\Omega)$ as an element of $L^2\left(\Omega_W; L^2(\Omega_J)\right)$. A smooth F then has form $F = \sum_{i=1}^n G_i H_i$ where each G_i is a smooth Brownian random variable and $H_i \in L^2(\Omega_J)$. We can then define D^W by $D^W F = \sum_{i=1}^n \left(D^W G_i\right) H_i$, where $D^W G_i$ is the classical Malliavin derivative. It can be shown that this definition can be extended to a subspace $\operatorname{Dom} D^W \subset \operatorname{Dom} D^0$, so that for $F \in \operatorname{Dom} D^W$, as expected,

$$D_{t,0} F = \frac{1}{\sigma} D_t^W F.$$

  For functionals of the form $F = f(G, H) \in L^2(\Omega)$ having $G \in \operatorname{Dom} D^W$, $H \in L^2(\Omega_J)$, and such that f is continuously differentiable with bounded partial derivatives in the first variable, we have a chain rule result: $F \in \operatorname{Dom} D^0$ and $D_{t,0} F = \frac{1}{\sigma} \frac{\partial f}{\partial x}(G, H) D_t^W G$. We may loosen the restriction on f to a.e. differentiability if G is absolutely continuous.

- The derivative D_z, $z=(t,x)$ with $x\neq 0$, is a difference operator: for $F\in \operatorname{Dom} D^J$

$$D_zF=\frac{F(\omega_{t,x})-F(\omega)}{x}$$

where, if Ψ_zF is the right-hand expression, then $\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}_0}(\Psi_zF)^2\,d\mu(z)\right]<\infty$. The idea is to introduce a jump of size x at time t which is captured by the realization $\omega_{t,x}$. For $\omega=(\omega^W,\omega^J)$, we define $\omega_{t,x}$ by simply adding the time-jump pair (t,x) to ω^J. For $F=f(G,H)\in L^2(\Omega)$ with $G\in L^2(\Omega_J)$, $H\in \operatorname{Dom} D^J$ and f continuous, we have this chain rule result:

$$D_zF=\frac{f(G,H(\omega_{t,x}))-f(G,H(\omega))}{x}=\frac{f(G,xD_zH+H(\omega))-f(G,H(\omega))}{x}.$$

If f is differentiable, then by the mean value theorem, for some random $\theta_z\in(0,1)$,

$$D_zF=\frac{\partial f}{\partial y}(G,\theta_zxD_zH+H(\omega))\,D_zH.$$

The following unified chain rule will be very useful (see Proposition 2 in [30]): If $F\in \operatorname{Dom} D^W\cap \operatorname{Dom} D^J$, $DF\in L^2_\mu$, $f\in\mathcal{C}^{k-1}$ has bounded first derivative (or f' may be unbounded if F is absolutely continuous with respect to Lebesgue measure) and $f^{(k-1)}$ is Lipschitz, then for $z\in(t,x)\in\mathbb{R}^+\times\mathbb{R}$,

$$D_zf(F)=\sum_{n=1}^{k-1}\frac{f^{(n)}(F)}{n!}x^{n-1}(D_zF)^n+\int_0^{D_zF}\frac{f^{(k)}(F+xu)}{(k-1)!}x^{k-1}(D_zF-u)^{k-1}du. \tag{6}$$

In the case where $f^{(k-1)}$ is differentiable everywhere, the chain rule is

$$D_zf(F)=\sum_{n=1}^{k-1}\frac{f^{(n)}(F)}{n!}x^{n-1}(D_zF)^n+\frac{f^{(k)}(F+\theta_zxD_zF)}{k!}x^{k-1}(D_zF)^k \tag{7}$$

for some function $\theta_z\in(0,1)$ for all $z=(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

We now define the adjoint of D (see [24] again). Suppose $F\in L^2(\mathbb{R}^+\times\mathbb{R}\times\Omega,\mathcal{B}(\mathbb{R}^+\times\mathbb{R})\times\mathcal{F},\mu\times\mathbb{P})$ with $F(z)=\sum_{q=0}^\infty I_q(f_q(z,\cdot))$ where each $f_q\in L^2_{\mu^{\otimes(q+1)}}$ is symmetric in the last q variables. In this case, the Skorohod integral of F is $\delta(F)=\sum_{q=0}^\infty I_{q+1}\left(\widetilde{f}_q\right)$ where $\sum_{q=0}^\infty(q+1)!\left\|\widetilde{f}_q\right\|^2_{L^2_{\mu^{\otimes(q+1)}}}<\infty$, i.e. $F\in\operatorname{Dom}\delta$ (by definition). Furthermore, $\mathbb{E}[\delta(F)G]=\mathbb{E}\left[\langle F,DG\rangle_{L^2_\mu}\right]$ for any $G\in\operatorname{Dom} D$.

Finally, we define as before $L=-\delta D$: for F as in (5), $LF=-\sum_{q=1}^\infty qI_q(f_q)$. The pseudo-inverse is defined by $L^{-1}F=-\sum_{q=1}^\infty\frac{1}{q}I_q(f_q)$. We have again $L^{-1}LF=F-\mathbb{E}[F]$.

Remark 1 *Write $\vec{z}_q=(z_1,\ldots,z_q)$, with $z_i=(t_i,x_i)$ for all i. Define*

$$\widetilde{W}=\left\{F=\sum_{q=0}^\infty I_q(f_q)\in\operatorname{Dom}D^0\ :\ f_q\in L^2_{\mu^{\otimes q}},\text{ and for every } q,\ f_q(\vec{z}_q)=0 \text{ if } x_i\neq 0 \text{ for some } i\right\}.$$

Notice from the previous discussion that if $f_q(\vec{z}_q)=0$ because $x_i\neq 0$, then $I_q(f_q)$ coincides with an iterated multiple (Wiener) integral. Therefore, Wiener space can be seen as a subspace of Wiener-Poisson space (similarly for Poisson space as a subspace). Moreover, $\widetilde{W}$ coincides with the subspace $\mathbb{D}^{1,2}$ (through embedding). The relevance of these facts is that if we have a r.v. $F\in\widetilde{W}$, then the chain rule formula and the Malliavin calculus operators are exactly (up to a constant) the same as those in Wiener space (as explained earlier in this subsection). Furthermore, the results (from other papers) in Wiener space can be replicated in $\widetilde{W}$ and so the conclusions will hold in Wiener-Poisson space, but within $\widetilde{W}$. From now on, $\mathbb{D}^{1,2}$ will mean the subspace $\mathbb{D}^{1,2}$ in Wiener space or the respective embedding $\widetilde{W}$ in Wiener-Poisson space.

2.3 The random variables g_X, g_Z and the functions g_*, G_*

From this point on, $\mathfrak{H}$ will be taken as $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}), \mu)$ if we are in Wiener-Poisson space. Now suppose F has mean 0. We have the following integration by parts formulas.

- If $F \in \operatorname{Dom} D^W \cap \operatorname{Dom} D^J$ and $f \in \mathcal{C}^1$ with Lipschitz first derivative assumed to be bounded (or f' may be unbounded if F has a density),

$$\mathbb{E}[Ff(F)] = \mathbb{E}\left[\left\langle -DL^{-1}F, DF\right\rangle_{\mathfrak{H}} f'(F)\right] + \mathbb{E}\left[\left\langle -DL^{-1}F, \int_0^{DF} f''(F+xu)x(DF-u)du\right\rangle_{\mathfrak{H}}\right]. \quad (8)$$

- If $F \in \operatorname{Dom} D^W \cap \operatorname{Dom} D^J$ and f is twice differentiable with bounded first derivative (or f' may be unbounded if F has a density),

$$\mathbb{E}[Ff(F)] = \mathbb{E}\left[\left\langle -DL^{-1}F, DF\right\rangle_{\mathfrak{H}} f'(F)\right] + \mathbb{E}\left[\left\langle -DL^{-1}F, \frac{f''(F+\theta.xDF)}{2}x(DF)^2\right\rangle_{\mathfrak{H}}\right]. \quad (9)$$

- If $F \in \mathbb{D}^{1,2}$ (see Remark 1) and f is differentiable with bounded derivative (or f is at least a.e. differentiable if F has a density),

$$\mathbb{E}[Ff(F)] = \mathbb{E}\left[\left\langle -DL^{-1}F, DF\right\rangle_{\mathfrak{H}} f'(F)\right]. \quad (10)$$

These formulas provide the link to using Malliavin calculus techniques in solving problems related to Stein's method. Since $F = LL^{-1}F = -\delta DL^{-1}F$, we have

$$\mathbb{E}[Ff(F)] = \mathbb{E}\left[-\delta DL^{-1}F \cdot f(F)\right] = \mathbb{E}\left[\left\langle -DL^{-1}F, Df(F)\right\rangle_{\mathfrak{H}}\right].$$

A direct application of the chain rule for Wiener-Poisson space, choosing $k=2$ in (6) and (7), yields (8) and (9) respectively, and an application of the respective chain rule in Wiener space yields (10).

Recall that in the Wiener-Poisson case (by Remark 1), if $F \in \mathbb{D}^{1,2}$ then the chain rule formula (7) will reduce to the corresponding one in Wiener space. Note that in this paper, we are assuming the target r.v. Z is in $\mathbb{D}^{1,2}$. Consequently, the points we will make about Z will be valid whether we are working in Wiener space or Wiener-Poisson space. The discussion of this subection then applies to both spaces.

Assumption A $Z \in \mathbb{D}^{1,2}$ has mean 0 and support (l,u) with $-\infty \leq l < 0 < u \leq \infty$. The density ρ_* of Z is known, and it is continuous in its support. X is either in $\mathbb{D}^{1,2}$ (Wiener space case) or in $\operatorname{Dom} D^W \cap \operatorname{Dom} D^J$ (Wiener-Poisson space case), and it also has mean 0.

Caution: Notice that in the previous subsection we used $x \in \mathbb{R}$ to denote the jump component of $z \in \mathbb{R}^+ \times \mathbb{R}$ in our state space. On the other hand, we are using Z to denote the target r.v. and X the r.v. with unknown distribution. A confusion may arise in the usage of x and X, or z and Z. However, we will stick with current notation for consistency with existing literature. In this regard, we urge the reader to keep in mind that x represents the size of the jump while X is a random variable not (directly) related to x. On the other hand, z is a jump (time of the jump, size of the jump) while Z is the target r.v. which has no jumps.

Remark 2 *In some results, we will consider instead of X a sequence $\{X_n\}$ of random variables. In this case, we have the same assumptions (and corresponding functionals, defined below) for each X_n. Note that for $Z \in \mathbb{D}^{1,2}$, the support necessarily has to be an interval (see Theorem 3.1 [18], Proposition 2.1.7 [19]), a consequence that carries over to Wiener-Poisson space. The continuity assumption of the density ρ_* is not strong at all, since general processes like solutions of stochastic differential equations driven by Brownian motion or (under mild conditions) fractional Brownian motion (for example see [2]) have continuous densities.*

Define the random variable $g_F = \mathbb{E}\left[\left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}\middle| F\right]$ for any Malliavin differentiable r.v. F (we will work with $F = Z$ and $F = X$ later). Nourdin and Peccati proved that $g_Z \geq 0$ almost surely (Proposition 3.9, [14]). Closely related is the function

$$g_*(z) := \mathbb{E}\left[\left\langle DZ, -DL^{-1}Z\right\rangle_{\mathfrak{H}}\middle| Z = z\right]. \tag{11}$$

Trivially, $g_Z = g_*(Z)$. Nourdin and Viens (Theorem 3.1 [18]) proved that Z has a density if and only if $g_*(Z) > 0$ a.s. Therefore, $g_*(Z) > 0$ a.s. (Assumption A) and $g_*(z) > 0$ for a.e. $z \in (l, u)$. Equation (10) implies $\mathbb{E}[Zf(Z)] = \mathbb{E}[g_Z f'(Z)]$. In the same manner, in Wiener space ($X \in \mathbb{D}^{1,2}$) we have $\mathbb{E}[Xf(X)] = \mathbb{E}[g_X f'(X)]$, while in Wiener-Poisson space ($X \in \operatorname{Dom} D^W \cap \operatorname{Dom} D^J$), this changes to $\mathbb{E}[Xf(X)] = \mathbb{E}[g_X f'(X)] + \mathbb{E}[\langle -DL^{-1}X, \frac{f''(X+\theta_z xDX)}{2}x(DX)^2\rangle_H]$. One needs to be careful not to write $\mathbb{E}[Xf(X)] = \mathbb{E}[g_*(X) f'(X)]$ or $g_*(X) = g_X$ a.s., both false, since g_* is derived from the law of Z (we would need the corresponding g_* of X to make it true).

Nourdin and Viens proved (same Theorem 3.1 [18]) that

$$g_*(z) = \frac{\int_z^u y\rho_*(y)\,dy}{\rho_*(z)} = -\frac{\int_l^z y\rho_*(y)\,dy}{\rho_*(z)} \qquad \text{for a.e. } z \in (l, u). \tag{12}$$

In their proof, they pointed out that $\varphi(z) := \int_z^u y\rho_*(y)\,dy = -\int_l^z y\rho_*(y)\,dy > 0$ for all $z \in (l, u)$. Since ρ_* is (necessarily) bounded (Assumption A), $\varphi(z)/\rho_*(z)$ is strictly positive (inside the support). From this point on, we will take g_* to be either (11) or the version (12), whichever suits our purposes. Furthermore, we can assume that $g_*(z) > 0$ for every (and not just for almost every) $z \in (l, u)$. Notice that using this definition of g_* we can conclude that $\left(g_*(z)\rho_*(z)\right)' = \varphi'(z) = -z\rho_*(z)$.

Given the density ρ_* of Z, we can compute g_* using (12). Some examples of known distributions with their g_* are given in Table 1. Recall that $g_*(z) = 0$ outside the support.

Conversely, one can retrieve the density ρ_* given g_* using the following noteworthy density formula Nourdin and Viens [18] proved:

$$\rho_*(z) = \frac{\mathbb{E}|Z|}{2g_*(z)} \exp\left(-\int_0^z \frac{y}{g_*(y)}dy\right). \tag{13}$$

Proposition 3 *g_* necessarily satisfies the following:*

$$\int_l^0 \frac{y}{g_*(y)}dy = -\infty \qquad \int_0^u \frac{y}{g_*(y)}dy = \infty. \tag{14}$$

Proof. With $\varphi(z)$ defined as before, Nourdin and Viens (Theorem 3.1 [18]) showed that

$$\int_0^z \frac{y}{g_*(y)}dy = \ln\frac{\varphi(0)}{\varphi(z)}.$$

Since $\varphi(z) \to 0$ as $z \to u$ and as $z \to l$, the result follows. ■

Remark 4 *The necessary conditions in Proposition 3 are actually not new. Stein (Lemma VI.3 [26]) has pointed out that these are necessary for (12) to hold.*

- *Suppose $g_*(x) = \alpha(x-l)^p$ for some constant $\alpha > 0$ and the support of Z is (l, ∞). Then $\int_0^\infty \frac{x}{g_*(x)}dx = \infty$ if and only if $p \leq 2$, and $\int_l^0 \frac{x}{g_*(x)} = -\infty$ if and only if $1 \leq p$. Similarly, if $g_*(x) = \alpha(u-x)^q$ over the support $(-\infty, u)$, $1 \leq q \leq 2$ necessarily. Also, if $g_*(x) = O(x^p)$ and the support is $(-\infty, \infty)$, then $p \leq 2$. If $g_*(x) = \alpha(u-x)^q(x-l)^p$ over the support (l, u), then $p \geq 1$ and $q \geq 1$ necessarily.*

- *Not every g_* satisfying (14) will belong to a random variable in $\mathbb{D}^{1,2}$. For instance, suppose Z is Inverse Gamma (see Table 1) having support (l,∞). For this random variable, $g_*(z) = \alpha(z-l)^2$ for some $\alpha > 0$. Z can be shown to have finite variance if and only if $\alpha < 1$. Thus, when $\alpha \geq 1$, (14) is satisfied but $Z \notin L^2(\Omega)$ so $Z \notin \mathbb{D}^{1,2}$. This means that the coverage of our method is not as extensive as we would like it to be. Hence, it is important that we check if the target r.v. belongs to $\mathbb{D}^{1,2}$.*

Z, with support and parameters	$\rho_*(z)$	$g_*(z)$
Normal $(-\infty,\infty)$: $\sigma > 0$	$\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{z^2}{2\sigma}\right)$	σ^2
Gamma (l,∞): $l=-rs$, $r>0$, $s>0$	$\frac{1}{s^r\Gamma(r)}(z-l)^{r-1}\exp\left(-\frac{z-l}{s}\right)$	$s(z-l)$
χ^2 (l,∞): $l=-v$, d.f. $v>0$	$\frac{1}{2^{v/2}\Gamma(v/2)}(z-l)^{\frac{v}{2}-1}\exp\left(-\frac{z-l}{2}\right)$	$2(z-l)$
Exponential (l,∞): $l=-\frac{1}{\lambda}$, $\lambda>0$	$\lambda\exp(-\lambda(z-l))$	$\frac{1}{\lambda}(z-l)$
Beta (l,u): $l=-\frac{r}{r+s}$, $u=1+l$, $r,s>0$	$\frac{1}{\beta(r,s)}(z-l)^{r-1}(1+l-z)^{s-1}$	$\frac{1}{r+s}(z-l)(1+l-z)$
Pearson Type IV $(-\infty,\infty)$: $t=-\frac{s}{2(r-1)}$, $r>\frac{3}{2}$	$C\left(1+(z-t)^2\right)^{-r}e^{s\arctan(z-t)}$	$\frac{1}{2(r-1)}\left(1+(z-t)^2\right)$
Student's T $(-\infty,\infty)$: d.f. $v>2$	$\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{z^2}{v}\right)^{-\frac{v+1}{2}}$	$\frac{v}{v-1}\left(1+\frac{z^2}{v}\right)$
Inverse Gamma (l,∞): $l=-\frac{s}{r-1}$, $r>3$, $s>0$	$\frac{s^{r-1}}{\Gamma(r-1)}(z-l)^{-r}\exp\left(-\frac{s}{z-l}\right)$	$\frac{1}{r-2}(z-l)^2$
Uniform (l,u): $u=-l>0$	$\frac{1}{2u}$	$\frac{1}{2}\left(u^2-z^2\right)$
Pareto (l,∞): $c>2$, $l<0$	$\frac{c(-l)^c(c-1)^c}{(z-cl)^{c+1}}$	$\frac{1}{c-1}(z-l)(z-cl)$
Laplace $(-\infty,\infty)$: $c>0$	$\frac{c}{2}\exp(-c\lvert z\rvert)$	$\frac{1}{c^2}(1+c\lvert z\rvert)$
Lognormal (l,∞): $l=-\exp\left(\delta+\frac{1}{2}\sigma^2\right)$	$\frac{-l}{\sqrt{2\pi}\sigma e^{2\delta}}\exp\left(-\frac{1}{2}[p(z)+\sigma]^2\right)$ where $p(z)=\frac{\ln(z-l)-\delta}{\sigma}$	$\sigma e^{2\delta}\exp\left(\frac{1}{2}(p(z)+\sigma)^2\right)$ $\times\int_{p(z)-\sigma}^{p(z)}e^{-s^2/2}ds$

Table 1: Common random variables Z with their ρ_* and g_*

Parallel to g_* of Z, we may define a corresponding object for X but we will have no use for it. In fact, we typically won't have access to the density of X; if it was known otherwise, one may characterize X without having to approximate it by Z. We will only assume properties (for X) amenable to the Malliavin calculus that would allow us to define g_X.

Let $G_*(z) = \int_l^z g_*(y)\,dy$ be the indefinite integral of g_* (assuming $g_* \in L^1(l,u)$). Consider the Wiener space case ($X \in \mathbb{D}^{1,2}$) and suppose $\|g_*\|_\infty < \infty$ or X has a density. If we take $f = G_*$ in (10), then

$$\mathbb{E}\left[g_X g_*(X)\right] = \mathbb{E}\left[G_*(X)\,X\right]. \tag{15}$$

Assumption A′ Along with Assumption A, either $\|g_*\|_\infty < \infty$ or X has a density.

Proposition 5 *(Moments formula)*

- **Wiener space:** $\mathbb{E}\left[F^{r+1}\right] = r\mathbb{E}\left[F^{r-1}g_F\right]$, *provided the expectations exist.*
  1. *If $g_*(Z) = g_Z$ is a polynomial in Z, i.e. $g_*(z) = \sum_{k=0}^m a_k z^k$, then* $\mathbb{E}\left[Z^{r+1}\right] = \sum_{k=0}^m r a_k \mathbb{E}\left[Z^{r+k-1}\right]$.
  2. *If $X = I_q(g)$, then* $\mathbb{E}\left[X^{r+1}\right] = \frac{r}{q}\mathbb{E}\left[X^{r-1}\|DX\|_{\mathfrak{H}}^2\right]$.
- **Wiener-Poisson space:** $\mathbb{E}\left[F^{r+1}\right] = r\mathbb{E}\left[F^{r-1}g_F\right] + \frac{r(r-1)}{2}\mathbb{E}\left[\langle -DL^{-1}F, x(F+\theta.xDF)^{r-2}(DF)^2\rangle_{\mathfrak{H}}\right]$
  1. *If $X = I_q(g)$, then* $\mathbb{E}\left[X^{r+1}\right] = \frac{r}{q}\mathbb{E}\left[X^{r-1}\|DX\|_{\mathfrak{H}}^2\right] + \frac{r(r-1)}{2q}\mathbb{E}\left[\left\langle x(DX)^3, (X+\theta.xDX)^{r-2}\right\rangle_{\mathfrak{H}}\right]$.

Proof. Simply let $f(y) = y^n$ in (9) and (10), taking into account the fact that $-DL^{-1}F = \frac{1}{q}DF$ when $F = I_q(g)$. ■

3 Stein's method and the Stein equation

Stein's method is a set of procedures that is often used to measure distances between random variables such as X and Z. More precisely, we're measuring the distance between the laws of X and Z. These distances take the form

$$d_{\mathcal{H}}(X,Z) = \sup_{h\in\mathcal{H}} \left|\mathbb{E}\left[h(X)\right] - \mathbb{E}\left[h(Z)\right]\right| \tag{16}$$

where $\mathcal{H}$ is a suitable family of functions. If we take $\mathcal{H}_W = \{h : \|h\|_L \le 1\}$ where $\|\cdot\|_L$ is the Lipschitz seminorm, then $d_W = d_{\mathcal{H}_W}$ is called Wasserstein distance. The bounded Wasserstein (Fortet-Mourier) distance corresponds to $\mathcal{H}_{FM} = \{h : \|h\|_L + \|h\|_\infty \le 1\}$. Clearly, $d_{FM} \le d_W$. d_{FM} is important because it metrizes convergence in distribution: $d_{FM}(X_n, Z) \to 0$ if and only if $X_n \overset{\text{Law}}{\longrightarrow} Z$. d_W on the other hand induces a topology stronger than that of convergence in distribution.

Nourdin and Peccati [14] mentioned other useful metrics. We have the Total Variation distance when $\mathcal{H}_{TV} = \{\mathbf{1}_B : B \text{ is Borel}\}$ and the Kolmogorov distance when $\mathcal{H}_K = \left\{\mathbf{1}_{(-\infty,z]} : z \in \mathbb{R}\right\}$. The latter for example is suited for the analysis of probability tails. However, in this paper, we will only consider d_W and d_{FM} as we try to find bounds for $d_{\mathcal{H}}(X,Z)$ by exploiting properties of Lipschitz functions $h \in \mathcal{H}$.

A Stein equation is at the root of Stein's method. Given Z and a test function h, the Stein equation is the differential equation

$$g_*(x)\,f'(x) - xf(x) = h(x) - \mathbb{E}\left[h(Z)\right] \tag{17}$$

having solution $f = f_h$. If the law of X is "close" to the law of Z, then we expect $\mathbb{E}\left[h(X)\right] - \mathbb{E}\left[h(Z)\right]$ to be close to 0, for h belonging to a large class of functions. Consequently, $\mathbb{E}\left[g_*(X)\,f'(X) - Xf(X)\right]$ would have to be close to 0. In fact, subject to certain technical conditions, the left-hand side of equation (17) provides a characterization of the law of Z: $\mathbb{E}\left[g_*(X)\,f'(X) - Xf(X)\right] = 0$ if and only if $X \overset{\text{Law}}{=} Z$ (in the equation, information about the law of Z is coded in g_*). The following proposition states this result in its precise form. For a quick proof, see Proposition 6.4 in [14]. The first statement is Lemma 1 in [26] by Stein.

Lemma 6 *(Stein's Lemma)*

1. *If f is continuous, piecewise continuously differentiable, and $\mathbb{E}\left[g_*\left(Z\right)\left|f'\left(Z\right)\right|\right]<\infty$, then*

$$\mathbb{E}\left[g_*\left(Z\right)f'\left(Z\right)-Zf\left(Z\right)\right]=0. \tag{18}$$

2. *If for every differentiable f, $x\mapsto\left|g_*\left(x\right)f'\left(x\right)\right|+\left|xf\left(x\right)\right|$ is bounded and*

$$\mathbb{E}\left[g_*\left(X\right)f'\left(X\right)-Xf\left(X\right)\right]=0, \tag{19}$$

then $X\overset{Law}{=}Z$.

Let $\mathcal{H}=\mathcal{H}_{FM}$ or $\mathcal{H}=\mathcal{H}_W$. Using (17) on (16), we have

$$d_{\mathcal{H}}\left(X,Z\right)\leq\sup_{f\in\mathfrak{F}_{\mathcal{H}}}\left|\mathbb{E}\left[g_*\left(X\right)f'\left(X\right)-Xf\left(X\right)\right]\right| \tag{20}$$

where the sup is taken over the family $\mathfrak{F}_{\mathcal{H}}$ of all Stein equation solutions f corresponding to $h\in\mathcal{H}$. Here the integration by parts formulas (8) and (10) allow us to rewrite the term $\mathbb{E}[Xf(X)]$ in terms of the derivatives of f and the r.v. g_X, as we pointed out before. For instance, in Wiener space,

$$d_{\mathcal{H}}\left(X,Z\right)\leq\sup_{f\in\mathfrak{F}_{\mathcal{H}}}\left|\mathbb{E}\left[g_*\left(X\right)f'\left(X\right)-g_Xf'\left(X\right)\right]\right|=\sup_{f\in\mathfrak{F}_{\mathcal{H}}}\left|\mathbb{E}\left[f'\left(X\right)\left(g_*\left(X\right)-g_X\right)\right]\right|. \tag{21}$$

Thus, to ensure that the distance between X and Z is small, $g_*\left(X\right)$ should be close to g_X. We also need to have a good control of $f'\left(X\right)$. One way of addressing this, taking note of Corollary 6.5 in [14], is by assuming a universal bound for $\mathbb{E}\left[f'\left(X\right)^2\right]$ for all $f\in\mathfrak{F}_{\mathcal{H}}$ since

$$d_{\mathcal{H}}\left(X,Z\right)\leq\sqrt{\sup_{f\in\mathfrak{F}_{\mathcal{H}}}\mathbb{E}\left[f'\left(X\right)^2\right]}\times\sqrt{\mathbb{E}\left[\left(g_*\left(X\right)-g_X\right)^2\right]}. \tag{22}$$

The first factor is intractable since it requires us to consider conditions on X in relation to all members f of the family $\mathfrak{F}_{\mathcal{H}}$. If however we have a uniform bound for f', then we can avoid imposing an additional restriction on X. In this case, we only need worry about how close $g_*\left(X\right)$ is to g_X in $L^2\left(\Omega\right)$. In fact, such a bound allows us to just consider how close $g_*\left(X\right)$ is to g_X in $L^1\left(\Omega\right)$. It is then interesting to see how information about the law of Z is contained in its Malliavin derivative. Questions of how close the law of X is to that of Z is passed on to how close $\mathbb{E}\left[\left\langle DZ,-DL^{-1}Z\right\rangle_{\mathfrak{H}}\middle|Z=X\right]$ is to $\mathbb{E}\left[\left\langle DX,-DL^{-1}X\right\rangle_{\mathfrak{H}}\middle|X\right]$. Notice though that this discussion needs to be modified slightly in Wiener-Poisson space, since the integration by parts formula (8) involves also the second derivative. Thus, we need to control (in a uniform way) both the first and second derivatives of the solution of the Stein equation. Due to this extra requirement, as will be seen later, we will not be able to apply our tools to as wide a scope of target r.v. Z, as we would be able to do in Wiener space.

3.1 Bound for f'

The Normal case in Wiener space:
If Z is standard Normal ($g_*(z)=1$), the Stein equation is $f'\left(x\right)-xf\left(x\right)=h\left(x\right)-\mathbb{E}\left[h\left(Z\right)\right]$ and it has solution $f\left(x\right)=e^{x^2/2}\int_{-\infty}^{x}\left[h\left(y\right)-\mathbb{E}\left[h\left(Z\right)\right]\right]e^{-y^2/2}dy$. Stein proved (Lemma II.3 in [26]) that $\left\|f'\right\|_\infty\leq 2\left\|h-\mathbb{E}\left[h\left(Z\right)\right]\right\|_\infty$. In fact, $\left\|f'\right\|_\infty\leq\min\left\{2\left\|h-\mathbb{E}\left[h\left(Z\right)\right]\right\|_\infty,4\left\|h'\right\|_\infty\right\}$ (see Lemma 2.3 [5]). For $h\in\mathcal{H}_{FM}$, $\left\|f'\right\|_\infty\leq 4$. It follows from (21) that $d_{FM}\left(X,Z\right)\leq k\mathbb{E}\left[\left|1-g_X\right|\right]\leq k\sqrt{\mathbb{E}\left[\left(1-g_X\right)^2\right]}$ with $k=4$. Similar

estimates for $h \in \mathcal{H}_W$ lead to a bound for d_W of the same form but with $k = 1$ (Lemma 4.2 [4], Lemma 1.2 [14]). How close the law of X is to the standard Normal law depends on how close g_X is to $g_Z = 1$ (in the L^1 sense).

In the general case, the Stein equation (17) has solution

$$f(x) = \frac{1}{g_*(x)\rho_*(x)} \int_l^x [h(y) - m_h]\rho_*(y)\,dy = \frac{-1}{g_*(x)\rho_*(x)} \int_x^u [h(y) - m_h]\rho_*(y)\,dy \tag{23}$$

for $x \in (l, u)$, where $m_h := \mathbb{E}[h(Z)]$.

The proof of the bound for f' when Z is Normal can be adapted to find a constant bound for $g_* f'$ in the non-Normal case. If g_* is uniformly bounded below by a positive number, we easily get a uniform bound for f'. Unfortunately, this is not always the case. In Table 1 we can see several examples of target r.v.'s for which g_* can get arbitrarily close to 0 in its support (for example, when Z is Gamma and $g_*(z) = s(z-l)_+$). Kusuoka and Tudor in [10] (Proposition 3) proved the following proposition to address this issue. We state it in the following form using notation and assumptions we have set.

Lemma 7 *Suppose we have the following conditions on g_*.*

1. *If $u < \infty$, then $\underline{\lim}_{x\to u} g_*(x)/(u-x) > 0$.*
2. *If $l > -\infty$, then $\underline{\lim}_{x\to l} g_*(x)/(x-l) > 0$.*
3. *If $u = \infty$, then $\underline{\lim}_{x\to u} g_*(x) > 0$.*
4. *If $l = -\infty$, then $\underline{\lim}_{x\to l} g_*(x) > 0$.*

Then the solution f of the Stein equation (17), for a given test function h with $\|h\|_\infty < \infty$ and $\|h'\|_\infty < \infty$, has derivative bounded as follows:

$$\|f'\|_\infty \le k\left(\|h\|_\infty + \|h'\|_\infty\right) \tag{24}$$

where the constant k depends on Z alone, and not on h.

Unfortunately, conditions **1** and **2** are too restrictive. Consider for instance a r.v. Z with support (l, ∞) and $g_*(x) = \alpha(x)(x-l)^p$, where $\alpha(x)$ is uniformly bounded below by some $\alpha_0 > 0$. From Remark 4, $1 \le p \le 2$ necessarily. Among all g_* of this form, Lemma 7 is thus only able to assure the needed boundedness of f' when $p = 1$. For instance, when Z is Inverse Gamma or Lognormal, condition **2** fails (see the corresponding g_* in Table 1). This stresses the need for less restrictive conditions on g_* that would allow us to include these cases and much more. The first requirement in order to achieve this is a good representation of the derivative f'.

Proposition 8 *For $x \in (l, u)$, the derivative f' of the solution of the Stein equation (17) is*

$$f'(x) = \frac{1}{g_*^2(x)\rho_*(x)} \int_x^u \int_l^x [1 - \Phi(s)]\,\Phi(t)\,[h'(t) - h'(s)]\,dtds.$$

where $\Phi(x) = \int_l^x \rho_*(t)dt$ is the cumulative distribution function of Z.

Proof. First,

$$\begin{aligned}
h(x)-m_h &= \int_l^x [h(x)-h(s)]\,\rho_*(s)\,ds + \int_x^u [h(x)-h(s)]\,\rho_*(s)\,ds \\
&= \int_l^x \left[\int_s^x h'(t)\,dt\right]\rho_*(s)\,ds - \int_x^u \left[\int_x^s h'(t)\,dt\right]\rho_*(s)\,ds \\
&= \int_l^x \left[\int_l^t \rho_*(s)\,ds\right] h'(t)\,dt - \int_x^u \left[\int_t^u \rho_*(s)\,ds\right] h'(t)\,dt \\
&= \int_l^x \Phi(t)\,h'(t)\,dt - \int_x^u [1-\Phi(t)]\,h'(t)\,dt
\end{aligned}$$

and so

$$\begin{aligned}
g_*(x)\,\rho_*(x)\,f(x) &= \int_l^x [h(y)-m_h]\,\rho_*(y)\,dy \\
&= \int_l^x \left[\int_l^y \Phi(t)\,h'(t)\,dt\right]\rho_*(y)\,dy - \int_l^x \left[\int_y^u [1-\Phi(t)]\,h'(t)\,dt\right]\rho_*(y)\,dy \\
&= \int_l^x \left[\int_t^x \rho_*(y)\,dy\right]\Phi(t)\,h'(t)\,dt \\
&\quad - \int_l^x \left[\int_l^t \rho_*(y)\,dy\right][1-\Phi(t)]\,h'(t)\,dt - \int_x^u \left[\int_l^x \rho_*(y)\,dy\right][1-\Phi(t)]\,h'(t)\,dt \\
&= \int_l^x [\Phi(x)-\Phi(t)]\,\Phi(t)\,h'(t)\,dt - \int_l^x \Phi(t)\,[1-\Phi(t)]\,h'(t)\,dt - \int_x^u \Phi(x)\,[1-\Phi(t)]\,h'(t)\,dt.
\end{aligned}$$

Cancelling some terms and solving for f,

$$f(x) = -\frac{1-\Phi(x)}{g_*(x)\,\rho_*(x)}\int_l^x \Phi(t)\,h'(t)\,dt - \frac{\Phi(x)}{g_*(x)\,\rho_*(x)}\int_x^u [1-\Phi(t)]\,h'(t)\,dt. \tag{25}$$

Observe that if $x<0$,

$$0=\mathbb{E}[Z]=\int_l^x t\rho_*(t)dt+\int_x^u t\rho_*(t)dt \le x\Phi(x)+g_*(x)\rho_*(x)$$

while if $x>0$,

$$0=\mathbb{E}[Z]=\int_l^x t\rho_*(t)dt+\int_x^u t\rho_*(t)dt \ge -g_*(x)\rho_*(x)+x[1-\Phi(x)].$$

Therefore, $0\le -x\Phi(x)\le g_*(x)\rho_*(x)\to 0$ as $x\to l$ and $0\le x[1-\Phi(x)]\le g_*(x)\rho_*(x)\to 0$ as $x\to u$. When we then integrate by parts,

$$\int_l^x \Phi(t)\,dt = t\Phi(t)\Big|_l^x - \int_l^x t\rho_*(t)\,dt = x\Phi(x)+g_*(x)\,\rho_*(x) \tag{26}$$

$$\int_x^u [1-\Phi(t)]\,dt = t[1-\Phi(t)]\Big|_x^u + \int_x^u t\rho_*(t)\,dt = -x[1-\Phi(x)]+g_*(x)\,\rho_*(x)\,. \tag{27}$$

Finally,

$$\begin{aligned}
g_*(x)\,f'(x) &= xf(x)+h(x)-m_h \\
&= \left(-\frac{x[1-\Phi(x)]}{g_*(x)\,\rho_*(x)}+1\right)\int_l^x \Phi(t)\,h'(t)\,dt - \left(\frac{x\Phi(x)}{g_*(x)\,\rho_*(x)}+1\right)\int_x^u [1-\Phi(t)]\,h'(t)\,dt \\
&= \frac{1}{g_*(x)\,\rho_*(x)}\int_x^u [1-\Phi(s)]\,ds\int_l^x \Phi(t)\,h'(t)\,dt - \frac{1}{g_*(x)\,\rho_*(x)}\int_l^x \Phi(t)\,dt\int_x^u [1-\Phi(s)]\,h'(s)\,ds
\end{aligned}$$

which leads to the given form of f'. $\blacksquare$

The bound (24) is not directly suited for d_W where we don't have a prescribed bound on $||h||_\infty$. A workaround, as pointed out in [10], is that for each $h \in \mathcal{H}_W$, we pass on the analysis to a sequence $\{h_n\}$ converging to h uniformly in every compact set, where $\{h_n\} \subset \left\{h \in C_0^1 : \|h'\|_\infty \le 1\right\}$. However, with the help of the previous lemma, we can overcome this complication by giving a bound for f' in terms of only $||h'||_\infty$. Recall that if h is Lipschitz, it is a.e. differentiable and $\|h'\|_\infty \le \|h\|_L$. Thus, the upper bound obtained here is immediately well suited for all $f \in \mathcal{F}_{FM}$ and for all $f \in \mathcal{F}_W$.

Theorem 9 *If applicable, assume conditions **3** and **4** from Lemma 7. Suppose there exists a positive function $\tilde{g} \in \mathcal{C}^1(l,u)$ such that*

1. $0 < \underline{\lim}_{x \to u}\, g_*(x)/\tilde{g}(x) \le \overline{\lim}_{x \to u}\, g_*(x)/\tilde{g}(x) < \infty$ *and* $\tilde{g}'(u^-) := \lim_{x \to u^-} \tilde{g}'(x) \in \boldsymbol{R}$ *exists.*[1]
2. $0 < \underline{\lim}_{x \to l}\, g_*(x)/\tilde{g}(x) \le \overline{\lim}_{x \to l}\, g_*(x)/\tilde{g}(x) < \infty$ *and* $\tilde{g}'(l^+) := \lim_{x \to l^+} \tilde{g}'(x) \in \boldsymbol{R}$ *exists.*

Then the solution f of the Stein equation (17), for a given test function h with $\|h'\|_\infty < \infty$, has derivative bounded as follows:

$$\|f'\|_\infty \le k\,\|h'\|_\infty \tag{28}$$

where the constant k depends on Z alone, and not on h.

Proof. First note that from Proposition 8,

$$|f'(x)| \le \frac{2\,\|h'\|_\infty}{g_*^2(x)\,\rho_*(x)} \int_x^u [1 - \Phi(s)]\,ds \int_l^x \Phi(t)\,dt. \tag{29}$$

Fix l' and u' s.t. $l < l' < 0 < u' < u$. Since $g_*(x)\rho_*(x)$ is continuous and strictly positive on $[l',u']$, it attains its minimum $m := \inf_{[l',u']} g_*(x)\rho_*(x) > 0$ on this compact set. Also by continuity of the density $M := \sup_{[l',u']} \rho_*(x) < \infty$, and $g_*(x) = \frac{g_*(x)\rho_*(x)}{\rho_*(x)} \ge \frac{m}{M} > 0$ on $[l',u']$, so $g_*^2(x)\rho_*(x) \ge \frac{m^2}{M}$. By the continuity and positivity of $I_1(x) := \int_x^u [1-\Phi(s)]\,ds$ and $I_2(x) := \int_l^x \Phi(t)\,dt$ we conclude that $K := \sup_{[l',u']} (I_1(x) \vee I_2(x)) < \infty$. By (29), $|f'(x)| \le \frac{2MK^2}{m^2}\|h'\|_\infty$ on $[l',u']$.

Since l' and u' were arbitrarily chosen, we only need to prove now that $\overline{\lim}_{x \to l} |f'(x)| \le k_1 ||h'||_\infty$ and $\overline{\lim}_{x \to u} |f'(x)| \le k_2 ||h'||_\infty$ for some finite constants k_1 and k_2. Due to the symmetry of the arguments it suffices to prove just one of these limits. Suppose l' was chosen small enough so that $\tilde{g} \in \mathcal{C}^1(l,l')$, and for some constants $0 < c \le C < \infty$, $c g_*(x) \le \tilde{g}(x) \le C g_*(x)$ on (l,l').

- **Case 1: $\boldsymbol{l > -\infty}$.**
  We show that the limit of the right-hand side of (29) is finite as $x \to l$. Note that in this case, $\int_x^u [1-\Phi(s)]\,ds = g_*(x)\,\rho_*(x) - x[1-\Phi(x)] \to |l|$. By L'Hôpital's rule,

$$\begin{aligned}\overline{\lim_{x\to l}}\,|f'(x)| &\le 2\,\|h'\|_\infty\,|l|\,\overline{\lim_{x\to l}} \frac{C\int_l^x \Phi(t)\,dt}{\tilde{g}(x)\,g_*(x)\,\rho_*(x)} \le 2\,\|h'\|_\infty\,|l|\,C\,\overline{\lim_{x\to l}} \frac{\Phi(x)}{-x\tilde{g}(x)\,\rho_*(x) + \tilde{g}'(x)\,g_*(x)\,\rho_*(x)} \\ &\le 2\,\|h'\|_\infty\,|l|\,C\,\overline{\lim_{x\to l}} \frac{\Phi(x)}{[-cx + \tilde{g}'(x)]\,g_*(x)\,\rho_*(x)} \le \frac{2\,\|h'\|_\infty\,|l|\,C}{\tilde{g}'(l^+) - cl}\,\overline{\lim_{x\to l}} \frac{\rho_*(x)}{-x\rho_*(x)} = \frac{2\,\|h'\|_\infty\,C}{\tilde{g}'(l^+) - cl}.\end{aligned}$$

  Since $\tilde{g}(l^+) := \lim_{z \to l^+} \tilde{g}_*(z) = 0$ and $\tilde{g} \ge 0$, we may assume l' is small enough so $\tilde{g}' \ge 0$ on (l,l'). Consequently, $\tilde{g}'(l^+) \ne cl < 0$.

[1] $\boldsymbol{R}$ stands for the extended real numbers, i.e. $\boldsymbol{R} = [-\infty, \infty]$.

- **Case 2: $l = -\infty$.**
Since $\underline{\lim}_{x \to -\infty} g_*(x) > 0$, we may suppose l' is small enough so that for some constant $m_0 > 0$, $g_*(x) \geq m_0$ over $(-\infty, l')$.

$$\begin{aligned}
\overline{\lim_{x \to -\infty}} |f'(x)| &\leq 2||h'||_\infty \overline{\lim_{x \to -\infty}} \frac{\left(g_*(x)\rho_*(x) - x[1 - \Phi(x)]\right)\int_{-\infty}^x \Phi(t)\,dt}{g_*^2(x)\rho_*(x)} \\
&\leq 2\|h'\|_\infty \left(\overline{\lim_{x \to -\infty}} \frac{\int_{-\infty}^x \Phi(t)\,dt}{m_0} + \overline{\lim_{x \to -\infty}} \frac{-x\int_{-\infty}^x \Phi(t)\,dt}{g_*^2(x)\rho_*(x)} \right) \\
&= 2\|h'\|_\infty \overline{\lim_{x \to -\infty}} \frac{|x|\int_{-\infty}^x \Phi(t)dt}{g_*^2(x)\rho_*(x)}
\end{aligned}$$

There are two subcases to consider depending on the behavior of $\tilde{g}(x)$ as $x \to -\infty$. From the continuity of $\tilde{g}$ and the existence of $\tilde{g}'(l^+)$, $L := \lim_{x \to -\infty} \tilde{g}(x)$ necessarily exists. If $L < \infty$, then $\lim_{x \to -\infty} \frac{\tilde{g}(x)}{|x|} = 0$. If $L = \infty$, then by L'Hôpital's rule, $\lim_{x \to -\infty} \frac{\tilde{g}(x)}{|x|} = -\lim_{x \to -\infty} \tilde{g}'(x) = -\tilde{g}'(l^+)$. In either case, $\lim_{x \to -\infty} \frac{\tilde{g}(x)}{|x|}$ exists.

  - **Subcase 1: $\lim_{x \to -\infty} \frac{\tilde{g}(x)}{|x|} = \infty$**
  Note that by (26), $\int_{-\infty}^x \Phi(t)dt = x\Phi(x) + g_*(x)\rho_*(x) \leq g_*(x)\rho_*(x)$ so

$$\frac{|x|\int_{-\infty}^x \Phi(t)dt}{g_*^2(x)\rho_*(x)} \leq C\frac{|x|g_*(x)\rho_*(x)}{\tilde{g}(x)g_*(x)\rho_*(x)} = C\frac{|x|}{\tilde{g}(x)}.$$

  Therefore

$$\overline{\lim_{x \to -\infty}} |f'(x)| \leq 2\|h'\|_\infty C \overline{\lim_{x \to -\infty}} \frac{|x|}{\tilde{g}(x)} = 0 < \infty.$$

  - **Subcase 2: $\lim_{x \to -\infty} \frac{\tilde{g}(x)}{|x|} < \infty$**
  Similarly from (26),

$$\frac{|x|\int_{-\infty}^x \Phi(t)dt}{g_*^2(x)\rho_*(x)} \leq \frac{\int_{-\infty}^x \frac{|x|}{|t|} g_*(t)\rho_*(t)dt}{m_0 g_*(x)\rho_*(x)} \leq \frac{\int_{-\infty}^x g_*(t)\rho_*(t)dt}{m_0 g_*(x)\rho_*(x)}.$$

  Therefore,

$$\begin{aligned}
\overline{\lim_{x \to -\infty}} |f'(x)| &\leq \frac{2||h'||_\infty}{m_0} \overline{\lim_{x \to -\infty}} \frac{\int_{-\infty}^x g_*(t)\rho_*(t)dt}{g_*(x)\rho_*(x)} \leq \frac{2||h'||_\infty}{m_0} \overline{\lim_{x \to -\infty}} \frac{g_*(x)\rho_*(x)}{-x\rho_*(x)} \\
&\leq \frac{2||h'||_\infty}{m_0} \overline{\lim_{x \to -\infty}} \frac{\tilde{g}(x)}{c|x|} < \infty.
\end{aligned}$$

The proof that $\overline{\lim}_{x \to u} |f'(x)| \leq k_2||h'||_\infty$ for some $k_2 < \infty$ is similar. ■

Note that if g_* is uniformly bounded below in a neighborhood of $l > -\infty$ (or for $u < \infty$) then condition **2** (**1** in the case of u) from Theorem 9 is not required (see discussion before Lemma 7). In the statement of the previous theorem, we can take $\tilde{g} = g_*$ if g_* is continuously differentiable (at least locally $\mathcal{C}^1$ close to the endpoints of the support), and in this case the conditions are trivially met. In other words, if we can check that $g_* \in \mathcal{C}^1(l, u)$ then bound (28) is automatically true (given the existence of $\tilde{g}'(u^-)$ and $\tilde{g}'(l^+)$). These new conditions are met by all r.v.'s in the Exponential family, Pearson family, and practically any other r.v. whose density is $\mathcal{C}^1$ and is strictly positive in its support. If g_* is not continuously differentiable, we can still get the bound but we are required to approximate g_* by a continuously differentiable function $\tilde{g}$ near the endpoints of the support. For example, consider the Laplace distribution where $g_*(x) = \frac{1}{c^2}(1 + c|x|)$ (see

Table 1). In this case g_* is differentiable everywhere except at 0. Therefore we can choose $\tilde{g}(x) = g_*(x)$ for all $x \in (-\infty, l') \cup (u', \infty)$ (with $-\infty < l' < 0 < u' < \infty$) and $\tilde{g}(x) = \phi(x)$ on (l', u') where ϕ is a smooth function such that $\tilde{g}$ is differentiable at l' and u'.

Assumption B We have the following conditions on g_*.

1. For some positive $\tilde{g} \in \mathcal{C}^1(l, u)$,
   (a) $0 < \underline{\lim}_{x\to u} g_*(x)/\tilde{g}(x) \le \overline{\lim}_{x\to u} g_*(x)/\tilde{g}(x) < \infty$.
   (b) $0 < \underline{\lim}_{x\to l} g_*(x)/\tilde{g}(x) \le \overline{\lim}_{x\to l} g_*(x)/\tilde{g}(x) < \infty$.
   (c) $\tilde{g}'(l^+)$ and $\tilde{g}'(u^-)$ exist.
2. If $u = \infty$, then $\underline{\lim}_{x\to u} g_*(x) > 0$.
3. If $l = -\infty$, then $\underline{\lim}_{x\to l} g_*(x) > 0$.

3.2 Bound for f''

For our convergence in distribution results in Wiener-Poisson space, we need a boundedness result for f''. The existence of f'' demands more conditions on g_* such as differentiability, which is understandable since we are requiring greater regularity in the solution of the Stein equation. In this setting, the existence of f'' will also immediately force most conditions of Theorem 9 to be satisfied. If we want to work with d_W or d_{FM}, we need to consider Lipschitz functions h, and for any such test function, we can only hope for it to be differentiable almost everywhere. Consequently, f'' must be understood in the almost everywhere sense, i.e., f'' is a version of the second derivative of f such that wherever the second derivative does not exist, f'' will have a value of 0.

Before setting out to find a bound, we point out the unfortunate fact that our results here will not apply to as wide a range of target r.v. Z as what happened for the first derivative. More specifically, we won't be able to give a finite bound for $|f''(x)|$ when $l > -\infty$ or $u < \infty$, as we were able to do for $|f'(x)|$ in Theorem 9. We actually have a counterexample to illustrate this: a r.v. Z with support $(l, \infty) \subsetneq \mathbb{R}$, such that for some Lipschitz and bounded h, $f''(x)$ does not tend to a finite limit as $x \to l$. A similar counterexample can be constructed for a r.v. Z with support $(-\infty, u) \subsetneq \mathbb{R}$, or with support $(l, u) \subsetneq \mathbb{R}$.

First, we make preliminary computations on f''. Differentiating (17) gives us the second derivative

$$f''(x) = \frac{x - g_*'(x)}{g_*(x)} f'(x) + \frac{1}{g_*(x)} f(x) + \frac{1}{g_*(x)} h'(x)$$

which, after considering the form of f in equation (25) and of f' given in Proposition 8, reduces to

$$f''(x) = \frac{A(x) \int_l^x \Phi(t) h'(t)\, dt + B(x) \int_x^u [1 - \Phi(s)] h'(s)\, ds + g_*^2(x) \rho_*(x) h'(x)}{g_*^3(x) \rho_*(x)} \tag{30}$$

where, with the help of (26) and (27),

$$\begin{aligned} A(x) &= (x - g_*'(x)) \int_x^u [1 - \Phi(s)]\, ds - g_*(x)(1 - \Phi(x)) \\ &= g_*(x) \rho_*(x)(x - g_*'(x)) - \left(x^2 - x g_*'(x) + g_*(x)\right)(1 - \Phi(x)) \end{aligned} \tag{31}$$

$$\begin{aligned} B(x) &= -(x - g_*'(x)) \int_l^x \Phi(t)\, dt - g_*(x) \Phi(x) \\ &= g_*(x) \rho_*(x)(g_*'(x) - x) - \left(x^2 - x g_*'(x) + g_*(x)\right) \Phi(x). \end{aligned} \tag{32}$$

Let $d(x) = g_*^3(x)\rho_*(x)$ and $n(x) = f''(x)d(x)$, the indicated denominator and numerator, respectively, of $f''(x)$. As $x \to l$, both $d(x)$ and $n(x)$ tend to 0. If h' happens to be differentiable, then by L'Hôpital's rule, $\lim_{x\to l} f''(x) = \lim_{x\to l} n'(x)/d'(x)$. It can be shown that $A'(x) = (2 - g_*''(x))\int_x^u [1-\Phi(s)]\,ds$ and $B'(x) = -(2 - g_*''(x))\int_l^x \Phi(t)\,dt$. Therefore

$$\begin{aligned}
n'(x) &= A'(x)\int_l^x \Phi(t)h'(t)\,dt + A(x)\Phi(x)h'(x) + B'(x)\int_x^u [1-\Phi(s)]h'(s)\,ds - B(x)[1-\Phi(x)]h'(x) \\
&\quad + [-xg_*(x)\rho_*(x) + g_*'(x)g_*(x)\rho_*(x)]h'(x) + g_*^2(x)\rho_*(x)h''(x) \\
&= (2-g_*''(x))\int_x^u [1-\Phi(s)]\,ds \int_l^x \Phi(t)h'(t)\,dt - (2-g_*''(x))\int_l^x \Phi(t)\,dt\int_x^u [1-\Phi(s)]h'(s)\,ds \\
&\quad + [A(x)\Phi(x) - B(x)(1-\Phi(x)) - (x-g_*'(x))g_*(x)\rho_*(x)]h'(x) + g_*^2(x)\rho_*(x)h''(x) \\
&= (2-g_*''(x))g_*^2(x)\rho_*(x)f'(x) + 0\cdot h'(x) + g_*^2(x)\rho_*(x)h''(x)
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{x\to l} f''(x) &= \lim_{x\to l}\frac{(2-g_*''(x))g_*^2(x)\rho_*(x)f'(x) + g_*^2(x)\rho_*(x)h''(x)}{(2g_*'(x)-x)g_*^2(x)\rho_*(x)} \\
&= \lim_{x\to l}\frac{2-g_*''(x)}{2g_*'(x)-x}f'(x) + \lim_{x\to l}\frac{h''(x)}{2g_*'(x)-x}.
\end{aligned}$$

Define the function $h(x) = \frac{4}{3}(x-l)^{3/2}$ on $(l,0)$, $h(x) = \frac{4}{3}|l|^{3/2}$ on $[0,\infty)$ and $h(x) = 0$ on $(-\infty, l]$. This function is clearly Lipschitz. Note that $h''(x) = \frac{1}{\sqrt{x-l}}$ on $(l,0)$. We now consider the same assumptions from Theorem 9 and see that $\overline{\lim}_{x\to l}|f'(x)| \le k\|h'\|_\infty$ and $\lim_{x\to l}\frac{h''(x)}{2g_*'(x)-x} = \infty$. We have thus found a Lipschitz function h for which $\lim_{x\to l}|f''(x)| = \infty$.

Remark 10 *From the above discussion we can't expect to have a universal bound on the second derivative of f unless the support of the target r.v. is $(-\infty,\infty)$. This is consistent with the known NP bound in Wiener-Poisson space developed in [30], where Z was Normal and hence had $(-\infty,\infty)$ for support. For the rest of this subsection, we will then assume that $l = -\infty$ and $u = \infty$.*

Theorem 11 *Assume that g_* is twice differentiable and $g_*''(x) < 2$. Suppose too that $\left|\frac{x-g_*'(x)}{x^2-xg_*'(x)+g_*(x)}\right|$ is bounded as $x \to -\infty$ and as $x \to \infty$. Then the solution f of the Stein equation (17), for a given test function h with $\|h'\|_\infty < \infty$, has second derivative bounded as follows:*

$$\|f''\|_\infty \le k\|h'\|_\infty \tag{33}$$

where the constant k depends on Z alone, and not on h.

Proof. Recall the functions A and B in (31) and (32). Using Lemma 7 in [8] (note that Φ there is defined as the upper probability tail), $A(x) \le 0$ and $B(x) \le 0$. Therefore, from (30),

$$\begin{aligned}
|f''(x)| &\le \frac{-A(x)}{g_*^3(x)\rho_*(x)}\int_l^x \Phi(t)\,dt\cdot\|h'\|_\infty + \frac{-B(x)}{g_*^3(x)\rho_*(x)}\int_x^u [1-\Phi(s)]\,ds\cdot\|h'\|_\infty + \frac{|h'(x)|}{g_*(x)} \\
\frac{g_*^3(x)\rho_*(x)|f''(x)|}{\|h'\|_\infty} &\le \left[-(x-g_*'(x))\int_x^u [1-\Phi(s)]\,ds + g_*(x)(1-\Phi(x))\right]\int_l^x \Phi(t)\,dt \\
&\quad + \left[(x-g_*'(x))\int_l^x \Phi(t)\,dt + g_*(x)\Phi(x)\right]\int_x^u [1-\Phi(s)]\,ds + g_*^2(x)\rho_*(x) \\
&= g_*(x)(1-\Phi(x))\int_l^x \Phi(t)\,dt + g_*(x)\Phi(x)\int_x^u [1-\Phi(s)]\,ds + g_*^2(x)\rho_*(x) \\
&= g_*(x)(1-\Phi(x))(g_*(x)\rho_*(x) + x\Phi(x)) + g_*(x)\Phi(x)(g_*(x)\rho_*(x) - x[1-\Phi(x)]) \\
&\quad + g_*^2(x)\rho_*(x) \\
&= 2g_*^2(x)\rho_*(x).
\end{aligned}$$

Due to the continuity of g_* and conditions of Assumption B when $l = -\infty$ and $u = \infty$, there is some $m_0 > 0$ such that $g_*(x) > m_0$ for all $x \in \mathbb{R}$. Then, $|f''(x)| \leq \frac{2||h'||_\infty}{g_*(x)} \leq \frac{2}{m_0}||h'||_\infty = k||h'||_\infty$. ■

One might think at first glance that the conditions of Theorem 11 are too restrictive. However, a closer look will show that they are all satisfied by members of the Pearson family having $(-\infty, \infty)$ as its support. Examples are the Pearson Type IV, Normal, and Student's T distributions (see Table 1 to check the conditions).

Assumption B′ Along with Assumption B, the following hold.

1. g_* is twice differentiable and $g''_* < 2$.
2. $\overline{\lim}_{x\to\pm\infty} \left| \frac{x - g'_*(x)}{x^2 - xg'_*(x) + g_*(x)} \right| < \infty$.

4 NP bound in Wiener space

From the results in subsection 3.1 all solutions of the Stein equation belong to the set $\mathcal{F}_\mathcal{H} = \{f \in \mathcal{C}^1(l,u) : ||f'||_\infty \leq k\}$, where the constant k depends on the distance $d_\mathcal{H}$ used (and so it implicitly depends on the set $\mathcal{H}$).

Theorem 12 *(NP bound) Let $d_\mathcal{H}$ be d_W or d_{FM}. Under Assumptions A and B,*

$$d_\mathcal{H}(X,Z) \leq k\mathbb{E}\left|g_*(X) - g_X\right| \leq k\mathbb{E}\left[\left(g_*(X) - g_X\right)^2\right]^{1/2} \tag{34}$$

$$\leq k\sqrt{\left|\mathbb{E}\left[g_*(X)^2\right] - \mathbb{E}\left[g_*(Z)^2\right]\right| + \left|\mathbb{E}\left[g_*(X)g_X\right] - \mathbb{E}\left[g_*(Z)g_Z\right]\right| + \left|\mathbb{E}\left[g_X^2\right] - \mathbb{E}\left[g_Z^2\right]\right|}. \tag{35}$$

Let $G_*(x)$ be an antiderivative of $g_*(x)$. Under Assumptions A′ and B,

$$d_\mathcal{H}(X,Z) \leq k\sqrt{\left|\mathbb{E}\left[g_*(X)^2\right] - \mathbb{E}\left[g_*(Z)^2\right]\right| + \left|\mathbb{E}\left[XG_*(X)\right] - \mathbb{E}\left[ZG_*(Z)\right]\right| + \left|\mathbb{E}\left[g_X^2\right] - \mathbb{E}\left[g_Z^2\right]\right|}. \tag{36}$$

In both statements, k is a finite constant depending only on Z and on $d_\mathcal{H}$.

Proof. The first bound in (34) follows from (21) and Theorem 9. The second bound follows from Hölder's Inequality. Let $\Delta = \mathbb{E}\left[\left(g_*(X) - g_X\right)^2\right]^{1/2}$. Since $\left(g_*(Z) - g_Z\right)^2 = 0$ a.s.,

$$\Delta^2 = \mathbb{E}\left[g_*(X)^2\right] - 2\mathbb{E}\left[g_*(X)g_X\right] + \mathbb{E}\left[g_X^2\right] - \left(\mathbb{E}\left[g_*(Z)^2\right] - 2\mathbb{E}\left[g_*(Z)g_Z\right] + \mathbb{E}\left[g_Z^2\right]\right)$$

and (35) follows. From (15) and Assumption A′ we have $\mathbb{E}\left[g_*(F)g_F\right] = \mathbb{E}\left[FG_*(F)\right]$, which proves (36). ■

The first inequality also follows from Theorem 1 and equation (19) in Kusuoka and Tudor [10]. The setup in their paper involves functions b and a. The function b is any function for which $\int_l^u b(x)\rho_*(x)\,dx = 0$ along with a few other mild conditions: $b > 0$ near l, $b < 0$ near u, $b\rho_*$ is continuous and bounded on (l,u). They then defined $a(x) = 2\int_l^x b(y)\rho_*(y)\,dy/\rho_*(x)$. Then for W a standard Brownian motion, the SDE

$$dY_t = b(Y_t)\,dt + \sqrt{a(Y_t)}dW_t \tag{37}$$

has a unique Markovian weak solution with invariant density ρ_*. With a and b as given above, from Theorem 1 in [10],

$$d_\mathcal{H}(X,Z) \leq k\mathbb{E}\left|\frac{a(X)}{2} - \left\langle DX, DL^{-1}\left\{b(X) - \mathbb{E}b(X)\right\}\right\rangle\right| + k\left|\mathbb{E}b(X)\right|. \tag{38}$$

If we take $b(x) = -x$, it follows that $a(x) = 2g_*(x)$. If X is centered, the right-hand side of (38) quickly reduces to $k\mathbb{E}|g_*(X) - g_X|$.

While the results in [10] appear more general, taking $b(x) = -x$ suffices. A careful analysis will reveal that the proofs of their main results depend only on the density ρ_* and the choice of b. While each choice of b arguably yields a different diffusion process Y, the invariant density is still ρ_*. Their analytical proofs are in fact independent of the stochastic differential equation (37) and the diffusion process arising from it. For this paper, we only need comparisons with the law of the reference variable Z. To this end, knowing the density ρ_* will suffice. The computations using $b(x) = -x$ and $a(x) = 2g_*(x)$ are much easier and this is reflected in the simplicity of (34) compared to (38).

Furthermore, as shown in the next theorem, the bounds we get from taking $b(x) = -x$ (see Theorem 12) are tight. Indeed, nothing is lost by choosing b this way.

Theorem 13 *(Law Characterization)* $X \overset{Law}{=} Z$ *if and only if all of the following are satisfied.*

1. $\mathbb{E}\left[g_*(X)^2\right] = \mathbb{E}\left[g_*(Z)^2\right]$
2. $\mathbb{E}[XG_*(X)] = \mathbb{E}[ZG_*(Z)]$
3. $\mathbb{E}\left[g_X^2\right] = \mathbb{E}\left[g_Z^2\right]$

Proof. If the three conditions are satisfied, Theorem 12 implies $d(X,Z) = 0$.

Now suppose $X \overset{\text{Law}}{=} Z$. They then have the same density ρ_* so **1** and **2** immediately follow. We next prove that $g_X \overset{\text{Law}}{=} g_Z$, imitating the technique Nourdin and Viens used to prove (12) (see Theorem 3.1 [18]). Let f be a continuous function with compact support, and F any antiderivative.

$$\begin{aligned}
\mathbb{E}[f(X)g_X] = \mathbb{E}[XF(X)] &= \int_l^u [x\rho_*(x)]F(x)\,dx \\
&= -F(x)\int_x^u y\rho_*(y)\,dy\Big|_{x\to l}^{x\to u} + \int_l^u f(x)\left[\int_x^u y\rho_*(y)\,dy\right]dx \\
&= \int_l^u f(x)\frac{\int_x^u y\rho_*(y)\,dy}{\rho_*(x)}\rho_*(x)\,dx = \mathbb{E}\left[f(X)\frac{\int_X^u y\rho_*(y)\,dy}{\rho_*(X)}\right]
\end{aligned}$$

so $g_X = \int_X^u y\rho_*(y)\,dy/\rho_*(X)$ a.s. This has the same distribution as $\int_Z^u y\rho_*(y)\,dy/\rho_*(Z)$, equal to g_Z a.s., so **3** then follows. ∎

Remark 14 *We see that* $\mathbb{E}\left[g_*(Z)^2\right] = \mathbb{E}[ZG_*(Z)] = \mathbb{E}\left[g_Z^2\right]$ *(see Lemma 5). Thus, for X to have the same law as Z, it is necessary and sufficient that* $\mathbb{E}\left[g_*(X)^2\right]$, $\mathbb{E}[XG_*(X)]$ *and* $\mathbb{E}\left[g_X^2\right]$ *(which a priori need not be all the same) are all equal to* $\mathbb{E}\left[g_Z^2\right]$. *The three conditions in Theorem 13 are stated in their current form due to the symmetry involved.*

That $\mathbb{E}\left[g_*(Z)^2\right] = \mathbb{E}[ZG_*(Z)] = \mathbb{E}\left[g_Z^2\right]$ are all equal depends on the specific structure of Z itself, and it is rooted in how g_* (and thus G_* as well) is defined in terms of the law of Z. Specifically, it is because $g_*(Z) = g_Z$ that we are able to use the integration by parts formula (10) on $g_*(Z)$. If we evaluate the function g_* at the random variable X , we cannot expect $g_*(X)$ to be equal to g_X because g_* is an object that "belongs" to Z. However, if X and Z are to be "almost" the same in law, we would expect X to "almost" satisfy the same relations/equations for Z, e.g. $\mathbb{E}\left[g_*(X)^2\right]$ " = "$\mathbb{E}[XG_*(X)]$. If g_* is a polynomial, then this amounts to checking that the moments of X satisfy the same conditions met by the moments of Z.

Granted, this method of moments is not sufficient. Hence, the need for condition **3**, $\mathbb{E}\left[g_X^2\right] = \mathbb{E}\left[g_Z^2\right]$, in Theorem 13.

The following versions of Theorem 13 and Theorem 12 for sequences are useful.

Corollary 15 *$X_n \to Z$ in distribution if all of the following are satisfied.*

1. $\mathbb{E}\left[g_*(X_n)^2\right] \to \mathbb{E}\left[g_*(Z)^2\right]$
2. $\mathbb{E}\left[g_*(X_n)\, g_{X_n}\right] \to \mathbb{E}\left[g_*(Z)\, g_Z\right]$ (under Assumption A)

   $\mathbb{E}\left[X_n G_*(X_n)\right] \to \mathbb{E}\left[Z G_*(Z)\right]$ (under Assumption A′)
3. $\mathbb{E}\left[g_{X_n}^2\right] \to \mathbb{E}\left[g_Z^2\right]$

Corollary 16 *$X_n \to Z$ in distribution if $g_*(X_n) - g_{X_n} \to 0$ in $L^1(\Omega)$*

Remark 17 *If we normalize so that $\operatorname{Var} X = \operatorname{Var} Z$, condition **3** in Theorem 13 can be replaced by $\operatorname{Var} g_X = \operatorname{Var} g_Z$ since $\mathbb{E}[g_X] = \operatorname{Var} X$. This also allows us to replace the term $\left|\mathbb{E}\left[g_X^2\right] - \mathbb{E}\left[g_Z^2\right]\right|$ in Theorem 12 by $\left|\operatorname{Var} g_X - \operatorname{Var} g_Z\right|$. In Corollary 15, we can replace condition **3** by $\operatorname{Var} g_{X_n} \to \operatorname{Var} g_Z$ if $\mathbb{E}\left[X_n^2\right] \to \mathbb{E}\left[Z^2\right]$.*

If Z is Normal with variance σ^2 so $g_*(y) = \sigma^2$, $G_*(y) = \sigma^2 y$ and $g_Z = \sigma^2$. If $\operatorname{Var} X = \sigma^2$, then

$$d_{\mathcal{H}}(X, Z) \le k\sqrt{\left|\sigma^4 - \sigma^4\right| + \sigma^2 \left|\mathbb{E}\left[X^2\right] - \mathbb{E}\left[Z^2\right]\right| + \left|\operatorname{Var} g_X - \operatorname{Var} g_Z\right|} = k\sqrt{\operatorname{Var} g_X} \tag{39}$$

where $k = 4$ if $d_{\mathcal{H}} = d_{FM}$ and $k = 1$ if $d_{\mathcal{H}} = d_W$. This retrieves Theorem 3.3 in [15]. If we have a bound on $\operatorname{Var} g_X$, this may be used to bound the distance. A Poincaré-type inequality may be used in this regard. See [17] (also for an explanation of the notation used below) where they use such a bound on $\operatorname{Var} g_X$ to get the following result:

$$d_{\mathcal{H}}(X, Z) \le \frac{k\sqrt{10}}{2\sigma} \left(\mathbb{E}\left[\left\|D^2 X \otimes_1 D^2 X\right\|_{\mathfrak{H}^{\otimes 2}}^2\right]\right)^{1/2} \left(\mathbb{E}\left[\left\|DX\right\|_{\mathfrak{H}}^4\right]\right)^{1/2}. \tag{40}$$

This was used in [17] and [30] to prove CLTs for functionals of Gaussian subordinated fields (applied to fBm and the solution of the O-U SDE driven by fBm, for all $H \in (0, 1)$).

4.1 Convergence when g_* is a polynomial

Many of the common random variables belong to the Pearson family of distributions, all of whose members are characterized by their g_* being polynomials of degree at most 2, i.e. $g_*(y) = \alpha y^2 + \beta y + \gamma$ in the support of Z. Some member distributions in this family are Normal (g_* is constant), Gamma (g_* has degree 1), Beta (g_* is quadratic with positive discriminant), Student's T-distribution (g_* is quadratic with negative discriminant) and Inverse Gamma (g_* is quadratic with zero discriminant).

Refer to [6] and [26] for more information about Pearson distributions, and [8] for Stein's method applied to comparisons of probability tails with a Pearson Z. From Remark 4, if the support of Z is unbounded and g_* is a polynomial, then Z is necessarily Pearson. If Z has bounded support and g_* is a polynomial, g_* may have degree exceeding 2 and in this case, Z is not Pearson.

Corollary 18 *If g_* is a polynomial $g_*(x) = \sum_{k=0}^m a_k x^k$, for the convergence $X_n \to Z$ in distribution, conditions **1** and **2** in Corollary 15 can be replaced by these conditions (respectively): $\mathbb{E}\left[X_n^k\right] \to \mathbb{E}\left[Z^k\right]$ for $k = 1, \ldots, 2m$, and $\mathbb{E}\left[X_n^k g_{X_n}\right] \to \mathbb{E}\left[Z^k g_Z\right]$ for $k = 1, \ldots, m$. Under assumption A′, the two conditions can be replaced by $\mathbb{E}\left[X_n^k\right] \to \mathbb{E}\left[Z^k\right]$ for $k = 1, \ldots, \max\{2m, m+2\}$.*

Proof. $g_*^2(x)$ has order $2m$ while $xG_*(x)$ has order $m+2$. The matching moments ensure condition **1** in Corollary 15 is satisfied, and under Assumption A$'$ also condition **2** is fulfilled. ■

Suppose $g_*(x) = \sum_{k=0}^{m} a_k x^k$. Note that

$$\mathbb{E}\left[g_*(Z)^2\right] = \mathbb{E}\left[\left(\sum_{k=0}^{m} a_k Z^k\right)^2\right] = \sum_{k=0}^{2m}\left(\sum_{i=0}^{k} a_i a_{k-i}\right)\mathbb{E}\left[Z^k\right]$$

while

$$\mathbb{E}\left[ZG_*(Z)\right] = \sum_{k=0}^{m} \frac{a_k}{k+1}\mathbb{E}\left[Z^{k+2}\right].$$

We noted earlier that $\mathbb{E}\left[g_*(Z)^2\right]$ and $\mathbb{E}\left[ZG_*(Z)\right]$ are equal. While the polynomial coefficients of the different moments of Z are different, and more moments may be involved in one expression compared to the other, the coefficients and the moments themselves should take care of this apparent difference to ensure equality under the expectation.

Suppose Z is Pearson with $g_Z = g_*(Z) = \alpha Z^2 + \beta Z + \gamma$. Using Lemma 5, we can prove the following recursive formula for the moments of Z: $\mathbb{E}\left[Z^{r+1}\right] = \frac{r\beta}{1-r\alpha}\mathbb{E}\left[Z^r\right] + \frac{r\gamma}{1-r\alpha}\mathbb{E}\left[Z^{r-1}\right]$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[g_Z\right] = \mathbb{E}\left[Z^2\right] &= \frac{\gamma}{1-\alpha} \\
2\mathbb{E}\left[Zg_Z\right] = \mathbb{E}\left[Z^3\right] &= \frac{2\beta\gamma}{(1-\alpha)(1-2\alpha)} \\
3\mathbb{E}\left[Z^2 g_Z\right] = \mathbb{E}\left[Z^4\right] &= \frac{6\beta^2\gamma + (1-2\alpha)3\gamma^2}{(1-\alpha)(1-2\alpha)(1-3\alpha)}
\end{aligned}$$

and

$$\mathbb{E}\left[g_Z^2\right] = \frac{\beta^2\gamma(1-\alpha) + \gamma^2(1-2\alpha)^2}{(1-\alpha)(1-2\alpha)(1-3\alpha)} \tag{41}$$

$$\operatorname{Var} g_Z = \mathbb{E}\left[g_*^2(Z)\right] - \left(\mathbb{E}\left[g_*(Z)\right]\right)^2 = \frac{\beta^2\gamma(1-\alpha)^2 + 2\alpha^2\gamma^2(1-2\alpha)}{(1-2\alpha)(1-3\alpha)(1-\alpha)^2}. \tag{42}$$

Corollary 19 *Suppose Z is a Pearson random variable and for the sequence $\{X_n\}$, $\operatorname{Var} X_n = \mathbb{E}[X_n^2] = \mathbb{E}[g_{X_n}] \to \frac{\gamma}{1-\alpha}$. The following are sufficient conditions so that $X_n \to Z$ in distribution.*

1. *When Z is Normal ($\alpha = \beta = 0$), $\operatorname{Var} g_{X_n} \to 0$.*
2. *When Z is Gamma ($\alpha = 0$), $\operatorname{Var} g_{X_n} \to \beta^2\gamma$ and*
   - *under Assumption A, $\mathbb{E}\left[X_n g_{X_n}\right] \to \beta\gamma$.*
   - *under Assumption A$'$, $2\mathbb{E}\left[X_n g_{X_n}\right] = \mathbb{E}\left[X_n^3\right] \to 2\beta\gamma$.*
3. *In the general case where $\alpha \neq 0$, $\operatorname{Var} g_{X_n}^2 \to \frac{\beta^2\gamma(1-\alpha)^2 + 2\alpha^2\gamma^2(1-2\alpha)}{(1-2\alpha)(1-3\alpha)(1-\alpha)^2}$ and*
   - *under Assumption A,*
     $2\mathbb{E}\left[X_n g_{X_n}\right], \mathbb{E}\left[X_n^3\right] \to \frac{2\beta\gamma}{(1-\alpha)(1-2\alpha)}$, and $3\mathbb{E}\left[X_n^2 g_{X_n}\right], \mathbb{E}\left[X_n^4\right] \to \frac{6\beta^2\gamma+(1-2\alpha)3\gamma^2}{(1-\alpha)(1-2\alpha)(1-3\alpha)}$.
   - *under Assumption A$'$,*
     $2\mathbb{E}\left[X_n g_{X_n}\right] = \mathbb{E}\left[X_n^3\right] \to \frac{2\beta\gamma}{(1-\alpha)(1-2\alpha)}$, and $3\mathbb{E}\left[X_n^2 g_{X_n}\right] = \mathbb{E}\left[X_n^4\right] \to \frac{6\beta^2\gamma+(1-2\alpha)3\gamma^2}{(1-\alpha)(1-2\alpha)(1-3\alpha)}$.

Proof. Apply Corollary 18 directly. ■

The first statement is the version for sequences of Corollary 3.4 in [18]. Alternatively, we could replace $\operatorname{Var} g_{X_n} \to 0$ by $\mathbb{E}\left[g_{X_n}^2\right] \to \gamma^2$. For the Gamma convergence, we can replace $\operatorname{Var} g_{X_n} \to \beta^2\gamma$ by $\mathbb{E}\left[g_{X_n}^2\right] \to \beta^2\gamma + \gamma^2$. When $\alpha \neq 0$, we can work with (41) instead of (42) so the statement will be in terms of $\mathbb{E}\left[g_{X_n}^2\right] \to \frac{\beta^2\gamma(1-\alpha)+\gamma^2(1-2\alpha)^2}{(1-\alpha)(1-2\alpha)(1-3\alpha)}$.

The next result follows from Corollary 16.

Corollary 20 *Suppose Z is a Pearson random variable. $X_n \to Z$ in distribution if $g_{X_n} - \alpha X_n^2 - \beta X_n \to \gamma$ in $L^1(\Omega)$.*

4.2 Convergence in a fixed Wiener chaos

When X is inside a fixed Wiener chaos so $X = I_q(f)$, we have more structure available. For example, $\left\langle DX, -DL^{-1}X\right\rangle_{\mathfrak{H}} = \frac{1}{q}\|DX\|_{\mathfrak{H}}^2$. Therefore, if $Z \overset{\text{Law}}{=} \mathcal{N}\left(0,\sigma^2\right)$ and $\mathbb{E}\left[\left(I_q(f)\right)^2\right] = \sigma^2$, (39) gives us the bound

$$d_{\mathcal{H}}(X,Z) \leq k\sqrt{\operatorname{Var} g_X} \leq k\sqrt{\operatorname{Var}\left(\frac{1}{q}\|DX\|_{\mathfrak{H}}^2\right)}.$$

One may then use bounds like

$$\operatorname{Var}\left(\frac{1}{q}\|DX\|_{\mathfrak{H}}^2\right) \overset{(a)}{=} \frac{1}{q^2}\mathbb{E}\left[\left(\|DX\|_{\mathfrak{H}}^2 - q\sigma^2\right)^2\right] \overset{(b)}{\leq} \frac{q-1}{3q}\left(\mathbb{E}\left[X^4\right] - 3\sigma^4\right) \tag{43}$$

to further cap the distance. Equality (a) follows from $\mathbb{E}\left[\frac{1}{q}\|DX\|_{\mathfrak{H}}\right] = \mathbb{E}[g_X] = \sigma^2$ and inequality (b) from Lemma 3.5 in [15]. These are quite important and known results which yield CLTs for functionals on a fixed Wiener chaos. For instance, if we have a sequence $\{X_n\} = \{I_q(f_n)\}$ where $\mathbb{E}\left[\left(I_q(f_n)\right)^2\right] \to \sigma^2$, then the following conditions are equivalent:

1. $X_n \to Z$ in distribution;
2. $\mathbb{E}\left[X_n^4\right] \to 3\sigma^4$;
3. $\|f_n \otimes_r f_n\|_{H^{\otimes(2q-2r)}} \to 0$ for all $r = 1, \ldots, q-1$;
4. $\|DX_n\|_{\mathfrak{H}}^2 \to q\sigma^2$ in $L^2(\Omega)$;
5. $\left\|D^2X_n \otimes_1 D^2X_n\right\|_{\mathfrak{H}^{\otimes 2}}^2 \to 0$ in $L^2(\Omega)$.

See [21] for (1) $\iff$ (2) $\iff$ (3), [20] for (1) $\iff$ (4), and [17] for (1) $\iff$ (5). These in some sense highlight the tightness of inequality (36) with the help of bounds like (40) and (43).

Corollary 21 *If $X_n = I_q(f_n)$ with $q \geq 1$, then condition 3 in Corollary 15 can be replaced by $\mathbb{E}\left[\|DX_n\|_{\mathfrak{H}}^4\right] \to q^2\mathbb{E}\left[g_*^2(Z)\right]$.*

Proof. This is a direct consequence of $\left\langle DX_n, -DL^{-1}X_n\right\rangle_{\mathfrak{H}} = \frac{1}{q}\|DX_n\|_{\mathfrak{H}}^2$ and $\mathbb{E}\left[g_Z^2\right] = \mathbb{E}\left[g_*^2(Z)\right]$. ■

From this and Corollary 20, we have the following result for the convergence in a fixed Wiener chaos to a Pearson random variable.

Corollary 22 *Let Z be Pearson with $g_*(z) = \alpha z^2 + \beta z + \gamma$ in its support. Fix $q \geq 2$. Suppose $X_n = I_q(f_n)$ and $\mathbb{E}\left[X_n^2\right] \to \frac{\gamma}{1-\alpha}$. If $\|DX_n\|_{\mathfrak{H}}^2 - q\alpha X_n^2 - q\beta X_n \to q\gamma$ in $L^1(\Omega)$, then $X_n \to Z$ in distribution.*

Remark 23 *Special cases of the above corollary are known results.*

- *Let Z be Normal with variance 1, i.e. $g_*(z) = 1$. Suppose $\mathbb{E}\left[X_n^2\right] \to 1$. Then $X_n \to Z$ in distribution if $\|DX_n\|_{\mathfrak{H}}^2 \to q$ in $L^2(\Omega)$. See [20].*
- *Let Z be Gamma with $g_*(z) = (2z + 2v)_+$, i.e. $\beta = 2$ and $\gamma = 2v$, where the parameters are chosen for consistency with the discussion in [16]. Suppose $\mathbb{E}\left[X_n^2\right] \to 2v$. Then $X_n \to Z$ in distribution if $\|DX_n\|_{\mathfrak{H}}^2 - 2qX_n \to 2qv$ in $L^2(\Omega)$.*

The result in the first item of this remark is known as the Nualart−Ortiz-Latorre criterion. In [28], the authors used it to prove that

$$C\sqrt{N}\ln(N)\left(\widehat{H}_N - H\right) \xrightarrow[N\to\infty]{} \mathcal{N}(0,1)$$

where $\widehat{H}_N$ is an estimator of the Hurst parameter H for fBm when $H \in \left(\frac{1}{2}, \frac{1}{3}\right)$ (see [28] for details).

4.3 Bilinear functionals of Gaussian subordinated fields

Let X_t be a centered Gaussian stationary process with covariance function $C(t) = \mathbb{E}[X_0 X_t] = \mathbb{E}[X_s X_{s+t}]$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ non-constant function such that with $Z \sim \mathcal{N}(0, C(0))$, $\mathbb{E}[|f(Z)|] < \infty$ and $\mathbb{E}\left[|f''(Z)|^4\right] < \infty$. Write $\mu_f = \mathbb{E}[f(Z)]$. It was proved in [17] and [30] (under mild conditions) that

$$H_T := \frac{1}{V(T)}\int_{[0,T]} (f(X_s) - \mu_f)\, ds \xrightarrow[T\to\infty]{} \mathcal{N}(0, \Sigma^2)$$

where $V(T)$ is a normalization function with specific properties. In particular, for the case of the increments of fBm ($X_t = B_{t+1}^H - B_t^H$) with Hurst parameter $H \in (1/2, 1)$, we have $V(T) = T^H$ and $\Sigma^2 = 2\big(\mathbb{E}[Zf(Z)]\big)^2$. As an ilustration of how to employ this tool, and to emphasize the advantage of using inequality (35) over (34), we will prove that $F_T := \left(\frac{H_T}{\Sigma}\right)^2 - \mathbb{E}\left[\left(\frac{H_T}{\Sigma}\right)^2\right]$ converges to a (centered) chi-squared r.v. as $T \to \infty$, for the case of the increments of fBm when $H \in (1/2, 1)$.

Preliminary computations and notation:
Following the setup in [19] (Section 5.1.3), we can write $B_{t+1}^H - B_t^H = I_1\big(K_H(t,\cdot)\big)$ where $K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_{s\vee t}^{t+1}(u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du\mathbf{1}_{[0,t+1]}$ and $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{1/2}$. The integral I_1 is with respect to a Wiener process W generating the same filtration as B^H, with the two processes related by $B_t = \int_0^t K_H(t,s)dW_s$.

Let's make the simplifying assumption $C(0) = 1$ so Z, $X_s \sim \mathcal{N}(0,1)$. To simplify notation, we will write $\int_{T^q}$ for $\int_{[0,T]^q}$ and $\mathbf{K}_t$ for $K_H(t,\cdot)$. Also let $\mathfrak{H} = L^2([0,T])$. Then $DX_t = \mathbf{K}_t$ and $\mathbb{E}[X_s X_t] = \mathbb{E}[I_1(\mathbf{K}_s)I_1(\mathbf{K}_t)] = \langle \mathbf{K}_s, \mathbf{K}_t\rangle_{\mathfrak{H}} = C(|s-t|) =: C_{st}$. Define for $T > 0$ and $s, t \in [0,T]$ the functionals

$$\begin{aligned}
&F_{st} = (f(X_s) - \mu_f)(f(X_t) - \mu_f)\\
&\widetilde{F}_T = \left(\frac{H_T}{\Sigma}\right)^2 = \frac{1}{\Sigma^2 T^{2H}}\int_{T^2}(f(X_s) - \mu_f)(f(X_t) - \mu_f)\, dsdt = \frac{1}{\Sigma^2 T^{2H}}\int_{T^2} F_{st}dsdt\\
&F_T = \widetilde{F}_T - \mathbb{E}\left[\widetilde{F}_T\right].
\end{aligned}$$

Let $\mathbf{s} = (s_1, \ldots, s_P) \in [0,T]^P$. We will use $\epsilon(s_i, s_j)$ to denote a nonnegative integer exponent indexed by a pair of variables from $\mathbf{s}$. Define

$$L(T) = \frac{1}{T^{PH}} \int_{T^N} \prod_{s_i \neq s_j} \left| C_{s_i s_j}^{\epsilon(s_i,s_j)} \right| d\mathbf{s} = \frac{1}{T^{PH}} \left\| \prod_{s_i \neq s_j} C_{s_i s_j}^{\epsilon(s_i,s_j)} \right\|_1$$

where inside the integral is the product of $Q = \binom{P}{2}$ factors. For example, if $P = 4$ and $\mathbf{s} = (s,t,u,v)$, then

$$L(T) = \frac{1}{T^{4H}} \int_{T^4} \left| C_{st}^{\epsilon(s,t)} C_{su}^{\epsilon(s,u)} C_{sv}^{\epsilon(s,v)} C_{tu}^{\epsilon(t,u)} C_{tv}^{\epsilon(t,v)} C_{uv}^{\epsilon(u,v)} \right| dsdtdudv. \tag{44}$$

Recall that $C_{st}^{\epsilon(s,t)} = \left(C(|s-t|)\right)^{\epsilon(s,t)}$, so the integration in (44) is being done only on the subscripts and not on the superscripts (since these are fixed exponents indexed only by the variables over which we're integrating).

Proposition 24 *With the previous notation,*

1. *For $q \geq 1$, take $c_q q! = \mathbb{E}[H_q(Z) f(Z)]$ where H_q is the q^{th} Hermite polynomial. Then,*

$$f(X_t) = \mu_f + \sum_{q=1}^{\infty} c_q I_q\left(\mathbf{K}_t^{\otimes q}\right).$$

2. *F_{st} has Wiener chaos decomposition*

$$\begin{aligned} F_{st} &= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} c_m c_{n-m} I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes(n-m)}\right) + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{r=1}^{\infty} d(m+r, n-m+r, r) I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes(n-m)}\right) C_{st}^r \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{r=0}^{\infty} e(m,n,r)\, d(m+r, n-m+r, r) I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes(n-m)}\right) C_{st}^r \end{aligned}$$

where $d(k,j,r) = c_k c_j r! \binom{k}{r}\binom{j}{r}$, and

$$e(m,n,r) = \begin{cases} 0 & \text{if } r = 0 \text{ and } m \in \{0,n\} \\ 1 & \text{otherwise} \end{cases}.$$

3. *For $0 \leq a,\, b \leq n$,*

$$\left\langle \mathbf{K}_s^{\otimes a} \widetilde{\otimes} \mathbf{K}_t^{\otimes(n-a)}, \mathbf{K}_u^{\otimes b} \widetilde{\otimes} \mathbf{K}_v^{\otimes(n-b)} \right\rangle_{\mathfrak{H}^{\otimes n}} = \frac{1}{\binom{n}{a}\binom{n}{b}} \sum_p \binom{n}{p, a-p, b-p, n-a-b+p} C_{su}^p C_{sv}^{a-p} C_{tu}^{b-p} C_{tv}^{n-a-b+p}$$

where the summation is taken over all p for which $\max(0, a+b-n) \leq p \leq \min(a,b)$.

4. *Fix an integer $P \geq 2$. Let $S = \sum \epsilon(s_i, s_j)$ be the sum of the exponents in $L(T)$.*
   - *If $S > P/2$, then $\lim_{T\to\infty} L(T) = 0$.*
   - *If $S = P/2$, then $\overline{\lim}_{T\to\infty} L(T) < \infty$.*

Proof. To prove the first point we expand f in terms of Hermite polynomials:

$$f(z) = c_0 + \sum_{q=1}^{\infty} c_q H_q(z)$$

with $c_q q! = \mathbb{E}\left[H_q\left(Z\right) f\left(Z\right)\right]$ for all $q \geq 0$. Since for any $h \in L^2([0,T])$ we have the relation $H_q(I_1(h)) = I_q(h^{\otimes q})$, then the result follows.

For the second point we have that $\mathbf{K}_t^{\otimes k} \otimes_r \mathbf{K}_s^{\otimes j} = \langle \mathbf{K}_t, \mathbf{K}_s \rangle_{\mathfrak{H}}^r \left(\mathbf{K}_t^{\otimes (k-r)} \otimes \mathbf{K}_s^{\otimes (j-r)} \right) = C_{ts}^r \left(\mathbf{K}_t^{\otimes (k-r)} \otimes \mathbf{K}_s^{\otimes (j-r)} \right)$. Therefore, from the previous point and the product formula (2),

$$\begin{aligned}
F_{st} &= \sum_{k=1}^{\infty} c_k H_k\left(X_s\right) \sum_{j=1}^{\infty} c_j H_j\left(X_t\right) = \sum_{k,j=1}^{\infty} c_k c_j I_k\left(\mathbf{K}_s^{\otimes k}\right) I_j\left(\mathbf{K}_t^{\otimes j}\right) \\
&= \sum_{k,j=1}^{\infty} c_k c_j \sum_{r=0}^{k\wedge j} r! \binom{k}{r}\binom{j}{r} I_{k+j-2r}\left(\mathbf{K}_s^{\otimes k} \otimes_r \mathbf{K}_t^{\otimes j}\right) = \sum_{k,j=1}^{\infty} c_k c_j \sum_{r=0}^{k\wedge j} r! \binom{k}{r}\binom{j}{r} C_{st}^r I_{k+j-2r}\left(\mathbf{K}_s^{\otimes (k-r)} \otimes \mathbf{K}_t^{\otimes (j-r)}\right).
\end{aligned}$$

Write $c_{st}\left(k,j,r\right) = d\left(k,j,r\right) C_{st}^r I_{k+j-2r}\left(\mathbf{K}_s^{\otimes (k-r)} \otimes \mathbf{K}_t^{\otimes (j-r)}\right)$ where $d\left(k,j,r\right) = c_k c_j r! \binom{k}{r}\binom{j}{r}$. Then

$$F_{st} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{k\wedge j} c_{st}\left(k,j,r\right) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_{st}\left(k,j,0\right) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=1}^{k\wedge j} c_{st}\left(k,j,r\right).$$

Applying Fubini's theorem for sums we have,

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_{st}\left(k,j,0\right) &= \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} c_m c_j I_{m+j}\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes j}\right) = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} c_m c_{n-m} I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes (n-m)}\right) \\
&= \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} c_m c_{n-m} I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes (n-m)}\right)
\end{aligned}$$

and

$$\sum_{j=1}^{\infty}\sum_{r=1}^{k\wedge j} = \sum_{j=1}^{k}\sum_{r=1}^{j} + \sum_{j=k+1}^{\infty}\sum_{r=1}^{k} = \sum_{r=1}^{k}\sum_{j=r}^{k} + \sum_{r=1}^{k}\sum_{j=k+1}^{\infty} = \sum_{r=1}^{k}\sum_{j=r}^{\infty}.$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=1}^{k\wedge j} c_{st}\left(k,j,r\right) &= \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{j=r}^{\infty} c_{st}\left(k,j,r\right) = \sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} c_{st}\left(m+r, n-m+r, r\right) \\
&= \sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{st}\left(m+r, n-m+r, r\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{r=1}^{\infty} d\left(m+r, n-m+r, r\right) C_{st}^r I_n\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes (n-m)}\right),
\end{aligned}$$

establishing the second point.

Point 3 requires counting the possible combinations in the inner product. Note first that the symmetric tensor product $\mathbf{K}_t^{\otimes a} \widetilde{\otimes} \mathbf{K}_s^{\otimes (n-a)}$ has $\binom{n}{a}$ distinct terms. Take any particular term α and list down all its n factors $\mathbf{K}_t$ and $\mathbf{K}_s$ in the order in which they appear. Now take any term β from $\mathbf{K}_u^{\otimes b} \widetilde{\otimes} \mathbf{K}_v^{\otimes (n-b)}$ and list down all its factors (in order) below those of α. Let p be the number of $(\mathbf{K}_t, \mathbf{K}_u)$ pairings. Thus, the number of pairings of the type $(\mathbf{K}_t, \mathbf{K}_v)$, $(\mathbf{K}_s, \mathbf{K}_u)$ and $(\mathbf{K}_s, \mathbf{K}_v)$ are $a-p$, $b-p$ and $n-a-b+p$, respectively. Finally, the number of pairs (α, β) which have p matching $\mathbf{K}_t$ and $\mathbf{K}_u$ is $\binom{n}{p, a-p, b-p, n-a-b+p}$.

Finally, to prove the fourth point we make use of Proposition 3 (point 4) in [30] which states that,

$$\int_{T^2} \left|C_{st}^{\epsilon}\right| dsdt = O\left(\frac{T^{2H}}{T^{(2-2H)(\epsilon-1)}}\right)$$

so

$$\|C^{\epsilon}_{st}\|_Q = \left[\int_{T^P} \left|C^{Q\epsilon}_{st}\right| d\mathbf{s}\right]^{1/Q} = \left[T^{P-2}\int_{T^2} \left|C^{Q\epsilon}_{st}\right| dsdt\right]^{1/Q} = O\left(\frac{T^{(P-2+2H)/Q}}{T^{(2-2H)(Q\epsilon-1)/Q}}\right).$$

By the generalized Hölder inequality for integrals,

$$\begin{aligned} L(T) &= \frac{1}{T^{PH}}\left\|\prod_{s_i\neq s_j} C^{\epsilon(s_i,s_j)}_{s_is_j}\right\|_1 \leq \frac{1}{T^{PH}}\prod_{s_i\neq s_j}\left\|C^{\epsilon(s_i,s_j)}_{s_is_j}\right\|_Q \\ &= O\left(\frac{1}{T^{PH}}\prod_{s_i\neq s_j}\frac{T^{(P-2+2H)/Q}}{T^{(2-2H)(Q\epsilon(s_i,s_j)-1)/Q}}\right) = O\left(\frac{1}{T^{(1-H)(2S-P)}}\right) \end{aligned}$$

and the result follows. ■

Theorem 25 *Suppose $f:\mathbb{R}\to\mathbb{R}$ is of class $\mathcal{C}^2$ such that $\Sigma^2 := 2\big(\mathbb{E}[Zf(Z)]\big)^2 \neq 0$. Then as $T\to\infty$,*

$$F_T \xrightarrow{Law} \chi^2.$$

Proof. Throughout this proof we will use the symbol $\sim$ to relate two expressions which have the same limit as $T\to\infty$. Also, $\boldsymbol{\kappa}$ will represent a constant whose value may change from one equation to another and might depend on the summation indices.

Since χ^2 is a member of the Pearson family, we can make use of Corollary 19 to prove this theorem. This will require proving convergence of the second moment of F_T and of g_{F_T}, plus the moment convergence of $F_T g_{F_T}$. We take note of the following facts: $\Sigma^2 = 2c_1^2$ ($c_1 = \mathbb{E}[Zf(Z)]$), and $T^{-2H}\int_{T^2} C_{ts}dtds \to 2$ as $T\to\infty$ (see Proposition 3 in [30]).

- **Convergence of the Second Moment of F_T:**
  From point 2 of Proposition 24,

$$DF_{st} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{\infty} ne(m,n,r)\,d(m+r,n-m+r,r)\,I_{n-1}\left(\mathbf{K}_s^{\otimes m}\otimes\mathbf{K}_t^{\otimes(n-m)}\right)C^r_{st} \tag{45}$$

  and

$$-DL^{-1}F_{uv} = \sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{R=0}^{\infty} e(M,N,R)\,d(M+R,N-M+R,R)\,I_{N-1}\left(\mathbf{K}_u^{\otimes M}\otimes\mathbf{K}_v^{\otimes(N-M)}\right)C^R_{uv}. \tag{46}$$

  Each indicated multiple integral of order $n-1$, with kernel in $\mathfrak{H}^{\otimes n}$, is to be interpreted as

$$\begin{aligned} I_{n-1}\left(\mathbf{K}_s^{\otimes m}\otimes\mathbf{K}_t^{\otimes(n-m)}\right) &= \frac{m}{n}I_{n-1}\left(\mathbf{K}_s^{\otimes(m-1)}\otimes\mathbf{K}_t^{\otimes(n-m)}\right)\mathbf{K}_s + \frac{n-m}{n}I_{n-1}\left(\mathbf{K}_s^{\otimes m}\otimes\mathbf{K}_t^{\otimes(n-m-1)}\right)\mathbf{K}_t \\ &= \sum_{a=m-1}^{m}\frac{k^{(m,n,a)}}{n}I_{n-1}\left(\mathbf{K}_s^{\otimes a}\otimes\mathbf{K}_t^{\otimes(n-1-a)}\right)\mathbf{f}^{(m,a)}_{st} \end{aligned} \tag{47}$$

  where $k^{(m,n,a)}\mathbf{f}^{(m,a)}_{st} = \begin{cases} m\mathbf{K}_s & \text{if } a=m-1\geq 0 \\ (n-m)\mathbf{K}_t & \text{if } a=m \end{cases}$.

Therefore, $\mathbb{E}\left[\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}\right]$ will be a summation containing terms of the form

$$\boldsymbol{\kappa}\mathbb{E}\left[\left\langle I_{n-1}\left(\mathbf{K}_s^{\otimes a}\otimes\mathbf{K}_t^{\otimes(n-1-a)}\right)\mathbf{f}_{st}^{(m,a)},I_{N-1}\left(\mathbf{K}_u^{\otimes A}\otimes\mathbf{K}_v^{\otimes(N-1-A)}\right)\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}}\right]C_{st}^rC_{uv}^R \tag{48}$$

$$=\mathbf{1}_{\{n=N\}}\boldsymbol{\kappa}\left\langle\mathbf{K}_s^{\otimes a}\widetilde{\otimes}\mathbf{K}_t^{\otimes(n-1-a)},\mathbf{K}_u^{\otimes A}\widetilde{\otimes}\mathbf{K}_v^{\otimes(n-1-A)}\right\rangle_{\mathfrak{H}^{\otimes(n-1)}}\left\langle\mathbf{f}_{st}^{(m,a)},\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}}C_{st}^rC_{uv}^R \tag{49}$$

$$=\mathbf{1}_{\{n=N\}}\sum_p\boldsymbol{\kappa}C_{su}^pC_{sv}^{a-p}C_{tu}^{A-p}C_{tv}^{n-1-a-A+p}\left\langle\mathbf{f}_{st}^{(m,a)},\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}}C_{st}^rC_{uv}^R \tag{50}$$

where from (49) to (50), we used the third point of Proposition 24.

Observe that $\left\langle\mathbf{f}_{st}^{(m,a)},\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}}$ is any of C_{su}, C_{sv}, C_{tu} or C_{tv}. In (50), the exponents of all six types of correlations then add up to $S=n+r+R=N+r+R$ (at this point, always take $N=n$, otherwise the term is 0). In the summations (45) and (46) appearing in $\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}$, since $e(m,n,r)\,e(M,N,R)=0$ if $S\leq 1$, the remaining terms are those for which $S\geq 2$. Therefore, using the moments formula of Proposition 5,

$$\mathbb{E}\left[F_T^2\right]=\mathbb{E}\left[g_{F_T}\right]=\mathbb{E}\left[\left\langle DF_T,-DL^{-1}F_T\right\rangle_{\mathfrak{H}}\right]=\frac{1}{\Sigma^4T^{4H}}\int_{T^4}\mathbb{E}\left[\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}\right]dsdtdudv \tag{51}$$

$$\sim\frac{1}{\Sigma^4T^{4H}}\int_{T^4}\mathbb{E}\left[\left\langle 2d(1,1,0)\,I_1(\mathbf{K}_s\otimes\mathbf{K}_t),d(1,1,0)\,I_1(\mathbf{K}_u\otimes\mathbf{K}_v)\right\rangle_{\mathfrak{H}}\right]dsdtdudv \tag{52}$$

where in (52), we applied point 4 of Proposition 24 ($P=4$) on (50) and (51): for those terms contributed by $\mathbb{E}\left[\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}\right]$ where $S>2$, the limit is 0. The limit in (51) is the nonzero value we get for the remaining case $S=2$; specifically, $(n,N,m,M,r,R)=(2,2,1,1,0,0)$. Since

$$\left\langle I_1(\mathbf{K}_s\otimes\mathbf{K}_t),I_1(\mathbf{K}_u\otimes\mathbf{K}_v)\right\rangle_{\mathfrak{H}}=\frac{I_1(\mathbf{K}_s)\,I_1(\mathbf{K}_u)\,C_{tv}+I_1(\mathbf{K}_t)\,I_1(\mathbf{K}_u)\,C_{sv}}{4}+\frac{I_1(\mathbf{K}_s)\,I_1(\mathbf{K}_v)\,C_{tu}+I_1(\mathbf{K}_t)\,I_1(\mathbf{K}_v)\,C_{su}}{4}, \tag{53}$$

then

$$\mathbb{E}\left[F_T^2\right]\sim\frac{2c_1^4}{\Sigma^4T^{4H}}\int_{T^4}\frac{1}{4}\left[2C_{su}C_{tv}+2C_{sv}C_{tu}\right]dsdtdudv\to\frac{2c_1^4}{\Sigma^4}(2)^2=2.$$

Therefore,

$$\boxed{\mathbb{E}\left[F_T^2\right]\to 2}$$

- **Convergence of $F_Tg_{F_T}$:**
Notice that

$$\mathbb{E}\left[F_Tg_{F_T}\right]=\mathbb{E}\left[\widetilde{F}_Tg_{F_T}\right]-\mathbb{E}\left[\widetilde{F}_T\right]\mathbb{E}\left[g_{F_T}\right]\sim\mathbb{E}\left[\widetilde{F}_Tg_{F_T}\right]-2$$

since

$$\mathbb{E}\left[\widetilde{F}_T\right]=\frac{1}{\Sigma^2T^{2H}}\int_{T^2}\mathbb{E}\left[F_{st}\right]dsdt=\sum_{r=1}^{\infty}\frac{d(r,r,r)}{\Sigma^2T^{2H}}\int_{T^2}C_{st}^rdsdt\sim\frac{d(1,1,1)}{\Sigma^2T^{2H}}\int_{T^2}C_{st}dsdt\to\frac{2c_1^2}{\Sigma^2}=1.$$

Now we need to investigate

$$\mathbb{E}\left[\widetilde{F}_Tg_{F_T}\right]=\frac{1}{\Sigma^6T^{6H}}\int_{T^6}\mathbb{E}\left[\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}F_{wx}\right]dsdtdudvdwdx. \tag{54}$$

The expression inside the expectation is a summation with generic term

$$
\begin{aligned}
&\left\langle I_{n-1}\left(\mathbf{K}_s^{\otimes m}\otimes \mathbf{K}_t^{\otimes(n-m)}\right)C_{st}^r, I_{N-1}\left(\mathbf{K}_u^{\otimes M}\otimes \mathbf{K}_v^{\otimes(N-M)}\right)C_{uv}^R\right\rangle_{\mathfrak{H}} I_{n'}\left(\mathbf{K}_w^{\otimes m'}\otimes \mathbf{K}_x^{\otimes\left(n'-m'\right)}\right)C_{wx}^{r'}\\
&\quad=\sum_{a,A}\boldsymbol{\kappa} I_{n-1}\left(\mathbf{K}_s^{\otimes a}\otimes \mathbf{K}_t^{\otimes(n-1-a)}\right)I_{N-1}\left(\mathbf{K}_u^{\otimes A}\otimes \mathbf{K}_v^{\otimes(N-1-A)}\right)I_{n'}\left(\mathbf{K}_w^{\otimes m'}\otimes \mathbf{K}_x^{\otimes\left(n'-m'\right)}\right)\left\langle \mathbf{f}_{st}^{(m,a)},\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}} C_{st}^r C_{uv}^R C_{wx}^{r'}
\end{aligned}
\tag{55}
$$

where $a\in\{m-1,m\}$, $A\in\{M-1,M\}$ and $\left\langle \mathbf{f}_{st}^{(m,a)},\mathbf{f}_{uv}^{(M,A)}\right\rangle_{\mathfrak{H}}$ could be any of C_{su}, C_{sv}, C_{tu} or C_{tv} (we used (47) here).

$$
\begin{aligned}
&\mathbb{E}\left[I_{n-1}\left(\mathbf{K}_s^{\otimes a}\otimes \mathbf{K}_t^{\otimes(n-1-a)}\right)I_{N-1}\left(\mathbf{K}_u^{\otimes A}\otimes \mathbf{K}_v^{\otimes(N-1-A)}\right)I_{n'}\left(\mathbf{K}_w^{\otimes m'}\otimes \mathbf{K}_x^{\otimes\left(n'-m'\right)}\right)\right]\\
&=\sum_{f=0}^{(n\wedge N)-1}\boldsymbol{\kappa}\left(\left[\left(\mathbf{K}_s^{\otimes a}\widetilde{\otimes}\mathbf{K}_t^{\otimes(n-1-a)}\right)\widetilde{\otimes}_f\left(\mathbf{K}_u^{\otimes A}\widetilde{\otimes}\mathbf{K}_v^{\otimes(N-1-A)}\right)\right]\widetilde{\otimes}_{n+N-2-2f}\left[\mathbf{K}_w^{\otimes m'}\widetilde{\otimes}\mathbf{K}_x^{\otimes\left(n'-m'\right)}\right]\right)\mathbf{1}_{\{n'=n+N-2-2f\}}\\
&=\sum_{\{2f=n+N-n'-2\}}\sum_{\{\epsilon\in\boldsymbol{A}_f\}}\boldsymbol{\kappa} C_{su}^{\epsilon(s,u)}C_{sv}^{\epsilon(s,v)}C_{tu}^{\epsilon(t,u)}C_{tv}^{\epsilon(t,v)}\\
&\qquad\times\left\langle \mathbf{K}_s^{\otimes(a-\epsilon(s,u)-\epsilon(s,v))}\widetilde{\otimes}\mathbf{K}_t^{\otimes(n-1-a-\epsilon(t,u)-\epsilon(t,v))}\widetilde{\otimes}\mathbf{K}_u^{\otimes(A-\epsilon(s,u)-\epsilon(t,u))}\widetilde{\otimes}\mathbf{K}_v^{\otimes(N-1-A-\epsilon(s,v)-\epsilon(t,v))},\mathbf{K}_w^{\otimes m'}\widetilde{\otimes}\mathbf{K}_x^{\otimes\left(n'-m'\right)}\right\rangle_{\mathfrak{H}^{\otimes n'}}\\
&=\sum_{\{2f=n+N-n'-2\}}\sum_{\{\epsilon\in\boldsymbol{A}_f,\xi\in\boldsymbol{B}_f^{\boldsymbol{A}}\}}\boldsymbol{\kappa} C_{su}^{\epsilon(s,u)}C_{sv}^{\epsilon(s,v)}C_{tu}^{\epsilon(t,u)}C_{tv}^{\epsilon(t,v)}C_{sw}^{\xi(s,w)}C_{sx}^{\xi(s,x)}C_{tw}^{\xi(t,w)}C_{tx}^{\xi(t,x)}C_{uw}^{\xi(u,w)}C_{ux}^{\xi(u,x)}C_{vw}^{\xi(v,w)}C_{vx}^{\xi(v,x)}
\end{aligned}
\tag{56}
$$

where $\boldsymbol{A}_f$ is the collection of exponents $\epsilon(t,u)=p$, $\epsilon(t,v)=S_1-p$, $\epsilon(s,u)=S_2-p$ and $\epsilon(s,v)=f-S_1-S_2+p$, with p, S_1, S_2 running over all integers such that $\max(0,S_1+S_2-f)\le p\le\min(S_1,S_2)$, $\max(0,n-1-a-f)\le S_1\le a$ and $\max(0,N-1-A-f)\le S_1\le A$. $\boldsymbol{B}_f^{\boldsymbol{A}}$ is defined similarly, just applying recursively point 3 of Proposition 24, so for instance we have $\xi(t,w)+\xi(t,x)+\xi(s,w)+\xi(s,x)+\xi(u,w)+\xi(u,x)+\xi(v,w)+\xi(v,x)=n'$. From this point on, we take $2f=n+N-n'-2$. Combining (55) and (56), we see that the expectation inside the integral of (54) is a summation

$$
\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{\infty}\sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{R=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{m'=0}^{n'}\sum_{r'=0}^{\infty}\sum_{a,A}\sum_{\{2f=n+N-n'-2\}}\sum_{\{\epsilon\in\boldsymbol{A}_f,\xi\in\boldsymbol{B}_f^{\boldsymbol{A}}\}}
$$

with each term consisting of 15 different types of correlations whose exponents add up to $S=f+n'+1+r+R+r'$. From this it follows that $n+N+n'+2\left(r+R+r'\right)=2S$. Systematically listing down all possible values of the indices in this summation will show that $e\left(m,n,r\right)e\left(M,N,R\right)e\left(m',n',r'\right)=0$ when $S\le 2$. For those terms in the summation for which $S>3$, the limit they contribute in (54) is 0 (using the fourth point in Proposition 24 with $P=6$ variables s, t, u, v, w, x). For those terms having $S=3$, a careful consideration of the indices such that $e\left(m,n,r\right)e\left(M,N,R\right)e\left(m',n',r'\right)>0$ leads to either $(m,n,r,M,N,R,m',n',r')=(1,2,0,1,2,0,0,0,1)$ or $(1,2,0,1,2,0,1,2,0)$. Therefore, we can continue (54) as

$$
\begin{aligned}
\mathbb{E}\left[\widetilde{F}_T g_{F_T}\right]&\sim\frac{2c_1^6}{\Sigma^6T^{6H}}\int_{T^6}\mathbb{E}\left[\left\langle I_1\left(\mathbf{K}_s\otimes\mathbf{K}_t\right),I_1\left(\mathbf{K}_u\otimes\mathbf{K}_v\right)\right\rangle_{\mathfrak{H}}\right]C_{wx}\,dsdtdudvdwdx\\
&\quad+\frac{2c_1^6}{\Sigma^6T^{6H}}\int_{T^6}\mathbb{E}\left[\left\langle I_1\left(\mathbf{K}_s\otimes\mathbf{K}_t\right),I_1\left(\mathbf{K}_u\otimes\mathbf{K}_v\right)\right\rangle_{\mathfrak{H}}I_2\left(\mathbf{K}_w\otimes\mathbf{K}_x\right)\right]dsdtdudvdwdx\\
&=L_1+L_2.
\end{aligned}
$$

Using (53), we have

$$L_1 = \frac{2c_1^6}{\Sigma^6} \cdot \frac{1}{T^{2H}} \int_{T^2} C_{wx} dwdx \cdot \frac{1}{T^{4H}} \int_{T^4} \mathbb{E}\left[\left\langle I_1\left(\mathbf{K}_s \otimes \mathbf{K}_t\right), I_1\left(\mathbf{K}_u \otimes \mathbf{K}_v\right)\right\rangle_{\mathfrak{H}}\right] dsdtdudv$$
$$\sim \frac{2c_1^6}{\Sigma^6} \cdot (2) \cdot \frac{1}{T^{4H}} \int_{T^4} \frac{2C_{su}C_{tv} + 2C_{sv}C_{tu}}{4} dsdtdudv \to \frac{2c_1^6}{\Sigma^6} \cdot (2) \cdot (2)^2 = 2\left(\frac{2c_1^2}{\Sigma^2}\right)^3 = 2.$$

To compute L_2, we use (53) again so

$$\begin{aligned} &\mathbb{E}\left[\left\langle I_1\left(\mathbf{K}_s \otimes \mathbf{K}_t\right), I_1\left(\mathbf{K}_u \otimes \mathbf{K}_v\right)\right\rangle_{\mathfrak{H}} I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] \\ &\quad = \frac{1}{4}\mathbb{E}\left[I_1\left(\mathbf{K}_s\right) I_1\left(\mathbf{K}_u\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] C_{tv} + \frac{1}{4}\mathbb{E}\left[I_1\left(\mathbf{K}_t\right) I_1\left(\mathbf{K}_u\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] C_{sv} \\ &\qquad + \frac{1}{4}\mathbb{E}\left[I_1\left(\mathbf{K}_s\right) I_1\left(\mathbf{K}_v\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] C_{tu} + \frac{1}{4}\mathbb{E}\left[I_1\left(\mathbf{K}_t\right) I_1\left(\mathbf{K}_v\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] C_{su}. \end{aligned}$$

The first expectation simplifies to

$$\begin{aligned} \mathbb{E}\left[I_1\left(\mathbf{K}_s\right) I_1\left(\mathbf{K}_u\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] &= \sum_{r=0}^{1} r! \binom{1}{r}\binom{1}{r} \mathbb{E}\left[I_{2-2r}\left(\mathbf{K}_s \otimes_r \mathbf{K}_u\right) I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] \\ &= 2! \left\langle \mathbf{K}_s \widetilde{\otimes} \mathbf{K}_u, \mathbf{K}_w \widetilde{\otimes} \mathbf{K}_x \right\rangle_{\mathfrak{H}^{\otimes 2}} = 2\frac{2C_{sw}C_{ux} + 2C_{sx}C_{uw}}{4} = C_{sw}C_{ux} + C_{sx}C_{uw} \end{aligned}$$

so

$$\begin{aligned} L_2 &= \frac{2c_1^6}{\Sigma^6 T^{6H}} \int_{T^6} \mathbb{E}\left[\left\langle I_1\left(\mathbf{K}_s \otimes \mathbf{K}_t\right), I_1\left(\mathbf{K}_u \otimes \mathbf{K}_v\right)\right\rangle_{\mathfrak{H}} I_2\left(\mathbf{K}_w \otimes \mathbf{K}_x\right)\right] dsdtdudvdwdx \\ &= \frac{2c_1^6}{4\Sigma^6 T^{6H}} \int_{T^6} \left[\begin{array}{c} \left(C_{sw}C_{ux} + C_{sx}C_{uw}\right) C_{tv} + \left(C_{tw}C_{ux} + C_{tx}C_{uw}\right) C_{sv} \\ + \left(C_{sw}C_{vx} + C_{sx}C_{vw}\right) C_{tu} + \left(C_{tw}C_{vx} + C_{tx}C_{vw}\right) C_{su} \end{array}\right] dsdtdudvdwdx \\ &= \frac{c_1^6}{2\Sigma^6 T^{6H}} \cdot 8 \int_{T^6} C_{sw}C_{ux}C_{tv} dsdtdudvdwdx \to \frac{4c_1^6}{\Sigma^6} (2)^3 = 4. \end{aligned}$$

Finally,

$$\boxed{\mathbb{E}\left[F_T g_{F_T}\right] \to (2+4) - 2 = 4.}$$

- **Convergence of the Second Moment of g_{F_T}:**

$$\begin{aligned} \mathbb{E}\left[g_{F_T}^2\right] &= \mathbb{E}\left[\left\langle DF_T, -DL^{-1}F_T\right\rangle_{\mathfrak{H}}^2\right] && (57) \\ &= \frac{1}{\Sigma^8 T^{8H}} \int_{T^8} \mathbb{E}\left[\left\langle DF_{st}, -DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}} \left\langle DF_{wx}, -DL^{-1}F_{yz}\right\rangle_{\mathfrak{H}}\right] dsdtdudvdwdxdydz. && (58) \end{aligned}$$

Inside the expectation, a generic term is

$$\begin{aligned} &\left\langle I_{n-1}\left(\mathbf{K}_s^{\otimes m} \otimes \mathbf{K}_t^{\otimes(n-m)}\right) C_{st}^r, I_{N-1}\left(\mathbf{K}_u^{\otimes M} \otimes \mathbf{K}_v^{\otimes(N-M)}\right) C_{uv}^R \right\rangle_{\mathfrak{H}} \\ &\qquad \times \left\langle I_{n'-1}\left(\mathbf{K}_w^{\otimes m'} \otimes \mathbf{K}_x^{\otimes\left(n'-m'\right)}\right) C_{wx}^{r'}, I_{N'-1}\left(\mathbf{K}_y^{\otimes M'} \otimes \mathbf{K}_z^{\otimes\left(N'-M'\right)}\right) C_{yz}^{R'} \right\rangle_{\mathfrak{H}} \\ &= \sum_{a,A,a',A'} \boldsymbol{\kappa} I_{n-1}\left(\mathbf{K}_s^{\otimes a} \otimes \mathbf{K}_t^{\otimes(n-1-a)}\right) I_{N-1}\left(\mathbf{K}_u^{\otimes A} \otimes \mathbf{K}_v^{\otimes(N-1-A)}\right) I_{n'-1}\left(\mathbf{K}_w^{\otimes a'} \otimes \mathbf{K}_x^{\otimes\left(n'-1-a'\right)}\right) \\ &\qquad \times I_{N'-1}\left(\mathbf{K}_y^{\otimes A'} \otimes \mathbf{K}_z^{\otimes\left(N'-1-A'\right)}\right) \left\langle \mathbf{f}_{st}^{(m,a)}, \mathbf{f}_{uv}^{(M,A)} \right\rangle_{\mathfrak{H}} \left\langle \mathbf{f}_{wx}^{\left(m',a'\right)}, \mathbf{f}_{yz}^{\left(M',A'\right)} \right\rangle_{\mathfrak{H}} C_{st}^r C_{uv}^R C_{wx}^{r'} C_{yz}^{R'}. \end{aligned}$$

The expectation of the product of the four multiple integrals is of the form

$$\mathbb{E}\left[I_{n-1}(a)\,I_{N-1}(b)\,I_{n'-1}(c)\,I_{N'-1}(d)\right]=\sum_{p,q}\boldsymbol{\kappa}\left\langle a\widetilde{\otimes}_p b, c\widetilde{\otimes}_q d\right\rangle_{\mathfrak{H}^{\otimes(n+N-2-2p)}}\mathbf{1}_{\{n+N-2p=n'+N'-2q\}}.$$

$\mathbb{E}\left[\left\langle DF_{st},-DL^{-1}F_{uv}\right\rangle_{\mathfrak{H}}\left\langle DF_{wx},-DL^{-1}F_{yz}\right\rangle_{\mathfrak{H}}\right]$ then is a summation

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{\infty}\sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{R=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{n'}\sum_{r'=0}^{\infty}\sum_{N'=1}^{\infty}\sum_{M'=0}^{N'}\sum_{R'=0}^{\infty}\sum_{a,A,a',A'}\sum_{\{n+N-2p=n'+N'-2q\}}$$

consisting of $\binom{8}{2}=28$ different types of correlations whose exponents add up to $S=p+q+(n+N-2-2p)+2+(r+R+r'+R')$. Along with the condition $n+N-2p=n'+N'-2q$, we have $2S=n+N+n'+N'+2(r+R+r'+R')$. By a careful consideration of the indices, $e(m,n,r)\,e(M,N,R)\,e(m',n',r')\,e(M',N',R')=0$ if $S\leq 3$. By point 4 in Proposition 24, using $P=8$ for the number of variables in (58), we see that the integrand terms for which $S>4$ contribute nothing to the limit as $T\to\infty$. If $S=4$ and $e(m,n,r)\,e(M,N,R)\,e(m',n',r')\,e(M',N',R')=1$, we only have $(m,n,r)=(M,N,R)=(m',n',r')=(M',N',R')=(1,2,0)$. Therefore,

$$\mathbb{E}\left[g_{F_T}^2\right]\sim\frac{4c_1^8}{\Sigma^8T^{8H}}\int_{T^8}\mathbb{E}\left[\left\langle I_1(\mathbf{K}_s\otimes\mathbf{K}_t),I_1(\mathbf{K}_u\otimes\mathbf{K}_v)\right\rangle_{\mathfrak{H}}\left\langle I_1(\mathbf{K}_w\otimes\mathbf{K}_x),I_1(\mathbf{K}_y\otimes\mathbf{K}_z)\right\rangle_{\mathfrak{H}}\right]dsdtdudvdwdxdydz.$$

We use (53) on the integrand:

$$\begin{aligned}&\mathbb{E}\left[\left\langle I_1(\mathbf{K}_s\otimes\mathbf{K}_t),I_1(\mathbf{K}_u\otimes\mathbf{K}_v)\right\rangle_{\mathfrak{H}}\left\langle I_1(\mathbf{K}_w\otimes\mathbf{K}_x),I_1(\mathbf{K}_y\otimes\mathbf{K}_z)\right\rangle_{\mathfrak{H}}\right]\\&=\frac{1}{16}\mathbb{E}\left[\begin{array}{c}\{I_1(\mathbf{K}_s)I_1(\mathbf{K}_u)C_{tv}+I_1(\mathbf{K}_t)I_1(\mathbf{K}_u)C_{sv}+I_1(\mathbf{K}_s)I_1(\mathbf{K}_v)C_{tu}+I_1(\mathbf{K}_t)I_1(\mathbf{K}_v)C_{su}\}\\\times\{I_1(\mathbf{K}_w)I_1(\mathbf{K}_y)C_{xz}+I_1(\mathbf{K}_x)I_1(\mathbf{K}_y)C_{wz}+I_1(\mathbf{K}_w)I_1(\mathbf{K}_z)C_{xy}+I_1(\mathbf{K}_x)I_1(\mathbf{K}_z)C_{wy}\}\end{array}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[g_{F_T}^2\right]&\sim\frac{4c_1^8}{\Sigma^8T^{8H}}\cdot\frac{1}{16}\cdot16\int_{T^8}\mathbb{E}\left[I_1(\mathbf{K}_s)I_1(\mathbf{K}_u)I_1(\mathbf{K}_w)I_1(\mathbf{K}_y)\right]C_{tv}C_{xz}dsdtdudvdwdxdydz\\&\sim\frac{4c_1^8}{\Sigma^8T^{4H}}\int_{T^4}\mathbb{E}\left[I_1(\mathbf{K}_s)I_1(\mathbf{K}_u)I_1(\mathbf{K}_w)I_1(\mathbf{K}_y)\right]dsdudwdy\cdot\frac{1}{T^{4H}}\int_{T^4}C_{tv}C_{xz}dtdvdxdz\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\left[I_1(\mathbf{K}_s)I_1(\mathbf{K}_u)I_1(\mathbf{K}_w)I_1(\mathbf{K}_y)\right]&=\sum_{p=0}^{1}\sum_{q=0}^{1}\mathbb{E}\left[I_{2-2p}(\mathbf{K}_s\otimes_p\mathbf{K}_u)\,I_{2-2q}(\mathbf{K}_w\otimes_p\mathbf{K}_y)\right]\\&=\mathbb{E}\left[I_2(\mathbf{K}_s\otimes\mathbf{K}_u)\,I_2(\mathbf{K}_w\otimes\mathbf{K}_y)\right]+\mathbb{E}\left[I_0(\mathbf{K}_s\otimes_1\mathbf{K}_u)\,I_0(\mathbf{K}_w\otimes_1\mathbf{K}_y)\right]\\&=2\left\langle\mathbf{K}_s\widetilde{\otimes}\mathbf{K}_u,\mathbf{K}_w\widetilde{\otimes}\mathbf{K}_y\right\rangle_{\mathfrak{H}^{\otimes2}}+C_{su}C_{wy}\\&=2\left\langle\frac{(\mathbf{K}_s\otimes\mathbf{K}_u)+(\mathbf{K}_u\otimes\mathbf{K}_s)}{2},\frac{(\mathbf{K}_w\otimes\mathbf{K}_y)+(\mathbf{K}_y\otimes\mathbf{K}_w)}{2}\right\rangle_{\mathfrak{H}^{\otimes2}}+C_{su}C_{wy}\\&=\frac{C_{sw}C_{uy}}{2}+\frac{C_{uw}C_{sy}}{2}+\frac{C_{sy}C_{uw}}{2}+\frac{C_{uy}C_{sw}}{2}+C_{su}C_{wy}\\&=C_{sw}C_{uy}+C_{uw}C_{sy}+C_{su}C_{wy}\end{aligned}$$

and so

$$\mathbb{E}\left[g_{F_T}^2\right]\sim\frac{4c_1^8}{\Sigma^8}\cdot\frac{1}{T^{4H}}\cdot3\int_{T^4}C_{sw}C_{uy}dsdudwdy\cdot\frac{1}{T^{4H}}\int_{T^4}C_{tv}C_{xz}dtdvdxdz\to\frac{4c_1^8}{\Sigma^8}\cdot3\,(2)^2\,(2)^2=12$$

proving that

$$\boxed{\mathbb{E}\left[g_{F_T}^2\right]\to12.}$$

Let $\beta = 2$ and $\gamma = 2$. We've shown that

$$\begin{aligned}\mathbb{E}\left[F_T^2\right] &\to 2 = \gamma \\ \mathbb{E}\left[F_T g_{F_T}\right] &\to 4 = \beta\gamma \\ \mathbb{E}\left[g_{F_T}^2\right] &\to 12 = \beta^2\gamma + \gamma^2.\end{aligned}$$

Therefore, F_T converges to a (centered) Gamma random variable. Specifically, since $\beta = 2$, it converges to a (centered) Chi-squared random variable with one degree of freedom (see Table 1). ■

Notice that this example is not trivial since is not a process in a fixed Wiener chaos. Also, it shows the importance of equation (36) in Theorem 12, since equation (34) is very intractable in this case.

5 NP bound in Wiener-Poisson space

In Wiener-Poisson space, if we repeat the process before equation (21) and use (8), the correct integration by parts formula, we get

$$d_{\mathcal{H}}(X,Z) \leq \sup_{f\in\mathcal{F}_{\mathcal{H}}} \left| \mathbb{E}\left[f'(X)\left(g_*(X) - g_X\right)\right] + \mathbb{E}\left[\left\langle \int_0^{D_z F} f''(F+xu)x(D_z F - u)du, -DL^{-1}X \right\rangle_{\mathfrak{H}}\right]\right| \quad (59)$$

It becomes evident that we need to find universal bounds on the first and second derivatives of f. Recall from subsection 3.2 that we only have such bounds when $l = -\infty$ and $u = \infty$. With this in mind, we have $\mathcal{F}_{\mathcal{H}} = \{f \in \mathcal{C}^1 : f' \text{ is Lipschitz}, ||f'||_\infty < k_1, ||f''||_\infty < k_2\}$, where k_1 and k_2 depend only on the distance $d_{\mathcal{H}}$. The following is a generalization of Theorem 2 in [30] (where Z was standard Normal) and an extension of Theorem 12 to Wiener-Poisson space.

Theorem 26 *(NP bound) Let $d_{\mathcal{H}}$ be d_W or d_{FM}. Under Assumptions A and B′,*

$$d_{\mathcal{H}}(X,Z) \leq k\Big(\mathbb{E}\left|g_*(X) - g_X\right| + \mathbb{E}\left[\left\langle \left|x\,(DX)^2\right|, \left|-DL^{-1}X\right|\right\rangle_{\mathfrak{H}}\right]\Big)$$

where k is a finite constant depending only on Z and on $d_{\mathcal{H}}$.

Proof. This follows immediately from (59) since $|\langle a,b\rangle| \leq \langle |a|,|b|\rangle$ and $\left|\int f\right| \leq \int |f|$. ■

This upper bound was first developed for Poisson space in [22], where was used to prove several CLTs for Poisson functionals. In [30] it was used to prove CLTs for Wiener-Poisson functionals.

Corollary 27 *$X_n \to Z$ in distribution if both statements are true.*

1. *$g_*(X_n) - g_{X_n} \to 0$ in $L^1(\Omega)$.*
2. *$\left\langle \left|x\,(DX_n)^2\right|, \left|-DL^{-1}X_n\right|\right\rangle_{\mathfrak{H}} \to 0$ in $L^1(\Omega)$.*

For convergence results inside a fixed Wiener chaos, the following preliminary computations are needed.

Proposition 28 *Let $X_n = I_q(f_n)$, with $\mathbb{E}\left[X_n^2\right] = q!\,\|f_n\|^2_{\mathfrak{H}^{\otimes q}} \to 1$. Assume that for $r = 0,\dots,q-1$ and $s = 0,\dots,q-r$, $\|f_n \otimes_r^s f_n\|_{\mathfrak{H}^{\otimes(2q-2r-s)}}\mathbf{1}_{\{s=0,r\neq0\}\cup\{s\neq0,r=0\}} \to 0$. Then as $n\to\infty$,*

1. *$\mathbb{E}\left[\|DX_n\|^4_{\mathfrak{H}}\right] \to q^2$;*

2. $||DX_n||_{\mathfrak{H}}^2 \to q$ *in* $L^2(\Omega)$*;*

3. $\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}} x^2 \left(D_z X_n\right)^4 d\mu\left(z\right)\right] \to 0$*;*

4. $\mathbb{E}\left[X_n^4\right] \to 3$.

Proof. Since $D_z X_n = qI_{q-1}\left(f_n\left(z,\cdot\right)\right)$, we can apply the product formula (4) to get

$$\begin{aligned}
\|DX_n\|_{\mathfrak{H}}^2 = \langle DX_n, DX_n\rangle_{\mathfrak{H}} &= q^2 \int I_{q-1}\left(f_n\left(z,\cdot\right)\right) I_{q-1}\left(f_n\left(z,\cdot\right)\right) d\mu\left(z\right) \\
&= q^2 \int \sum_{r=0}^{q-1} \sum_{s=0}^{q-1-r} r!s! \binom{q-1}{r}^2 \binom{q-1-r}{s}^2 I_{2q-2-2r-s}\big(f_n\left(z,\cdot\right) \otimes_r^s f_n\left(z,\cdot\right)\big) d\mu\left(z\right) \\
&= q^2 \sum_{p=1}^{q} \sum_{s=0}^{q-p} (p-1)!s! \binom{q-1}{p-1}^2 \binom{q-p}{s}^2 \int I_{2q-2p-s}\left(f_n\left(z,\cdot\right) \otimes_{p-1}^s f_n\left(z,\cdot\right)\right) d\mu\left(z\right) \\
&= \sum_{p=1}^{q} \sum_{s=0}^{q-p} pp!s! \binom{q}{p}^2 \binom{q-p}{s}^2 I_{2q-2p-s}\left(\int f_n\left(z,\cdot\right) \otimes_{p-1}^s f_n\left(z,\cdot\right) d\mu\left(z\right)\right) \\
&= \sum_{r=1}^{q} \sum_{s=0}^{q-r} rr!s! \binom{q}{r}^2 \binom{q-r}{s}^2 I_{2q-2r-s}\left(f_n \otimes_r^s f_n\right).
\end{aligned}$$

Also by orthogonality of chaoses,

$$\begin{aligned}
\mathbb{E}\left[\|DX_n\|_{\mathfrak{H}}^4\right] &= \sum_{r,R=1}^{q} \sum_{s=0}^{q-r} \sum_{S=0}^{q-R} rRr!R!s!S! \binom{q}{r}^2 \binom{q}{R}^2 \binom{q-r}{s}^2 \binom{q-R}{S}^2 \mathbb{E}\left[I_{2q-2r-s}\left(f_n \otimes_r^s f_n\right) I_{2q-2R-S}\left(f_n \otimes_R^S f_n\right)\right] \\
&\overset{(*)}{\leq} \sum_{\substack{r,R=1\\ r\neq R}}^{q} \sum_{s=0}^{q-r} \sum_{S=0}^{q-R} rRr!R!s!S! \binom{q}{r}^2 \binom{q}{R}^2 \binom{q-r}{s}^2 \binom{q-R}{S}^2 \\
&\qquad \times \mathbf{1}_{\{2r+s=2R+S\}} \left(2q-2r-s\right)! \left|\left|f_n \otimes_r^s f_n\right|\right|_{\mathfrak{H}^{\otimes(2q-2r-s)}} \left|\left|f_n \otimes_R^S f_n\right|\right|_{\mathfrak{H}^{\otimes(2q-2r-s)}} \\
&\quad + \sum_{r=1}^{q-1} \sum_{s=0}^{q-r} r^2 \left(r!\right)^2 \left(s!\right)^2 \binom{q}{r}^4 \binom{q-r}{s}^4 \left(2q-2r-s\right)! \left\|f_n \otimes_r^s f_n\right\|_{\mathfrak{H}^{\otimes(2q-2r-s)}}^2 + q^2 \left(q! \left\|f_n\right\|_{\mathfrak{H}^{\otimes q}}^2\right)^2
\end{aligned}$$

In (*), we used $||\widetilde{g}||_{\mathfrak{H}} \leq ||g||_{\mathfrak{H}}$ for nonsymmetric g (this follows by a simple application of the triangle inequality), and Hölder's inequality in the following:

$$\begin{aligned}
\mathbb{E}\left[I_{2q-2r-s}\left(f_n \otimes_r^s f_n\right) I_{2q-2R-S}\left(f_n \otimes_R^S f_n\right)\right] &= \mathbf{1}_{\{2r+s=2R+S\}} \left(2q-2r-s\right)! \left\langle f_n \widetilde{\otimes}_r^s f_n, f_n \widetilde{\otimes}_R^S f_n\right\rangle_{\mathfrak{H}^{\otimes(2q-2r-s)}} \\
&\leq \mathbf{1}_{\{2r+s=2R+S\}} \left(2q-2r-s\right)! \left|\left|f_n \widetilde{\otimes}_r^s f_n\right|\right|_{\mathfrak{H}^{\otimes(2q-2r-s)}} \left|\left|f_n \widetilde{\otimes}_R^S f_n\right|\right|_{\mathfrak{H}^{\otimes(2q-2r-s)}}.
\end{aligned}$$

$\mathbb{E}\left[\|DX_n\|_{\mathfrak{H}}^4\right] \to q^2$ then follows from the assumptions on the kernels' contractions, proving the first point.

On the other hand,

$$\mathbb{E}\left[\left(\|DX_n\|_{\mathfrak{H}}^2 - q\right)^2\right] = \mathbb{E}\left[\|DX_n\|_{\mathfrak{H}}^4 - 2q\left\|DX_n\right\|_{\mathfrak{H}}^2 + q^2\right] = \mathbb{E}\left[\|DX_n\|_{\mathfrak{H}}^4\right] - 2q \cdot q\mathbb{E}\left[X_n^2\right] + q^2 \to 0$$

so $\|DX_n\|_{\mathfrak{H}}^2 \to q$ in $L^2\left(\Omega\right)$ proving the second point.

For the third point we have,

$$
\begin{aligned}
(D_z X_n)^2 &= q^2 \sum_{r=0}^{q-1} \sum_{s=0}^{q-1-r} r!s! \binom{q-1}{r}^2 \binom{q-1-r}{s}^2 I_{2q-2-2r-s}\big(f_n(z,\cdot) \otimes_r^s f_n(z,\cdot)\big) \\
(D_z X_n)^4 &= q^4 \sum_{r=0}^{q-1} \sum_{R=0}^{q-1} \sum_{s=0}^{q-1-r} \sum_{S=0}^{q-1-R} r!R!s!S! \binom{q-1}{r}^2 \binom{q-1}{R}^2 \binom{q-1-r}{s}^2 \binom{q-1-R}{S}^2 \\
&\qquad \times I_{2q-2-2r-s}\big(f_n(z,\cdot) \otimes_r^s f_n(z,\cdot)\big) I_{2q-2-2R-S}\left(f_n(z,\cdot) \otimes_R^S f_n(z,\cdot)\right) \\
\mathbb{E}\left[\int x^2 (D_z X_n)^4 \, d\mu(z)\right] &= q^4 \sum_{r=0}^{q-1} \sum_{R=0}^{q-1} \sum_{s=0}^{q-1-r} \sum_{S=0}^{q-1-R} r!R!s!S! \binom{q-1}{r}^2 \binom{q-1}{R}^2 \binom{q-1-r}{s}^2 \binom{q-1-R}{S}^2 \\
&\qquad \times \int \mathbb{E}\left[I_{2q-2-2r-s}\big(xf_n(z,\cdot) \otimes_r^s f_n(z,\cdot)\big) I_{2q-2-2R-S}\left(xf_n(z,\cdot) \otimes_R^S f_n(z,\cdot)\right)\right] d\mu(z).
\end{aligned}
$$

The expectation, when $2r+s = 2R+S$, is bounded by

$$
\begin{aligned}
&(2q-2r-s-2)! \left| \left\langle xf_n(z,\cdot) \widetilde{\otimes}_r^s f_n(z,\cdot), xf_n(z,\cdot) \widetilde{\otimes}_R^S f_n(z,\cdot) \right\rangle_{\mathfrak{H}^{\otimes(2q-2R-S-2)}} \right| \\
&\qquad \le (2q-2r-s-2)! \left\| xf_n(z,\cdot) \widetilde{\otimes}_r^s f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}} \left\| xf_n(z,\cdot) \widetilde{\otimes}_R^S f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2R-S-2)}}.
\end{aligned}
$$

Modulo the constant factor $(2q-2r-s-2)!$, the integral of the expectation is bounded by

$$
\begin{aligned}
&\int \left\| xf_n(z,\cdot) \widetilde{\otimes}_r^s f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}} \left\| xf_n(z,\cdot) \widetilde{\otimes}_R^S f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2R-S-2)}} d\mu(z) \\
&\qquad = \left\langle \left\| xf_n(z,\cdot) \widetilde{\otimes}_r^s f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}}, \left\| xf_n(z,\cdot) \widetilde{\otimes}_R^S f_n(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2R-S-2)}} \right\rangle_{\mathfrak{H}} \\
&\qquad \le \left\| \| xf_n(z,\cdot) \otimes_r^s f_n \|_{H^{\otimes(2q-2r-s-2)}} \right\|_{\mathfrak{H}} \times \left\| \| xf_n(z,\cdot) \otimes_R^S f_n(z,\cdot) \|_{\mathfrak{H}^{\otimes(2q-2R-S-2)}} \right\|_{\mathfrak{H}}
\end{aligned}
$$

We'll work out the first factor:

$$
\begin{aligned}
\left\| \| xf_n(z,\cdot) \otimes_r^s f_n(z,\cdot) \|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}} \right\|_{\mathfrak{H}}^2 &= \int \| xf_n(z,\cdot) \otimes_r^s f_n(z,\cdot) \|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}}^2 \, d\mu(z) \\
&= \int \left\| \left(f_n \otimes_r^{s+1} f_n\right)(z,\cdot) \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-2)}}^2 d\mu(z) = \left\| f_n \otimes_r^{s+1} f_n \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-1)}}^2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbb{E}\left[\int x^2 (D_z X_n)^4 \, d\mu(z)\right] &\le q^4 \sum_{r,R=0}^{q-1} \sum_{s=0}^{q-1-r} \sum_{S=0}^{q-1-R} r!R!s!S! \binom{q-1}{r}^2 \binom{q-1}{R}^2 \binom{q-1-r}{s}^2 \binom{q-1-R}{S}^2 \\
&\qquad \times \mathbf{1}_{\{2r+s=2R+S\}} (2q-2r-s-2)! \left\| f_n \otimes_r^{s+1} f_n \right\|_{\mathfrak{H}^{\otimes(2q-2r-s-1)}}^2 \left\| f_n \otimes_R^{S+1} f_n \right\|_{\mathfrak{H}^{\otimes(2q-2R-S-1)}}^2 \\
&= q^4 \sum_{r,R=0}^{q-1} \sum_{t=1}^{q-r} \sum_{T=1}^{q-R} r!R!(t-1)!(T-1)! \binom{q-1}{r}^2 \binom{q-1}{R}^2 \binom{q-1-r}{t-1}^2 \binom{q-1-R}{T-1}^2 \\
&\qquad \times \mathbf{1}_{\{2r+t=2R+T\}} (2q-2r-t-1)! \left\| f_n \otimes_r^t f_n \right\|_{\mathfrak{H}^{\otimes(2q-2r-t)}}^2 \times \left\| f_n \otimes_R^T f_n \right\|_{\mathfrak{H}^{\otimes(2q-2R-T)}}^2
\end{aligned}
$$

The third point then follows.

Finally, for the fourth point, we use the moments formula Proposition 5 to get

$$
\mathbb{E}\left[X_n^4\right] = \frac{3}{q} \mathbb{E}\left[X_n^2 \|DX_n\|_H^2\right] + \frac{3}{q} \mathbb{E}\left[\left\langle x(DX_n)^3, X_n + \theta_z x D X_n \right\rangle_{\mathfrak{H}}\right] = \frac{3}{q} U_n + \frac{3}{q}(V_n + W_n)
$$

where

$$V_n = \mathbb{E}\left[\left\langle x\left(DX_n\right)^3, X_n\right\rangle_{\mathfrak{H}}\right] = \mathbb{E}\left[\left\langle x\left(DX_n\right)^2, X_n\left(DX_n\right)\right\rangle_{\mathfrak{H}}\right]$$

$$W_n = \mathbb{E}\left[\left\langle x\left(DX_n\right)^3, \theta_z x DX_n\right\rangle_{\mathfrak{H}}\right] = \mathbb{E}\left[\int \theta_z x^2\left(D_z X_n\right)^4 d\mu\left(z\right)\right]$$

and $U_n = \mathbb{E}\left[X_n^2 \left\|DX_n\right\|_{\mathfrak{H}}^2\right]$. It is sufficient to prove that $U_n \to q$, $V_n \to 0$ and $W_n \to 0$ as $n \to \infty$.

To compute U_n, note that $X_n^2 = \sum_{r=0}^q \sum_{s=0}^{q-r} r!s!\binom{q}{r}^2\binom{q-r}{s}^2 I_{2q-2r-s}\left(f_n \otimes_r^s f_n\right)$. Using our expression for $\left\|DX_n\right\|_{\mathfrak{H}}^2$ above,

$$\begin{aligned}
U_n &= \sum_{r=0}^{q}\sum_{R=1}^{q}\sum_{s=0}^{q-r}\sum_{S=0}^{q-R} Rr!R!s!S!\binom{q}{r}^2\binom{q}{R}^2\binom{q-r}{s}^2\binom{q-R}{S}^2 \mathbb{E}\left[I_{2q-2r-s}\left(f_n \otimes_r^s f_n\right) I_{2q-2R-S}\left(f_n \otimes_R^S f_n\right)\right] \\
&= \sum_{r=0}^{q}\sum_{R=1}^{q}\sum_{s=0}^{q-r}\sum_{S=0}^{q-R} Rr!R!s!S!\binom{q}{r}^2\binom{q}{R}^2\binom{q-r}{s}^2\binom{q-R}{S}^2 \mathbf{1}_{\{2r+s=2R+S\}}\left(2q-2r-s\right)!\left\langle f_n\widetilde{\otimes}_r^s f_n, f_n\widetilde{\otimes}_R^S f_n\right\rangle_{\mathfrak{H}^{\otimes(2q-2r-s)}} \\
&= \sum_{\substack{r=0,R=1\\ r\neq R}}^{q}\sum_{s=0}^{q-r}\sum_{S=0}^{q-R} Rr!R!s!S!\binom{q}{r}^2\binom{q}{R}^2\binom{q-r}{s}^2\binom{q-R}{S}^2 \mathbf{1}_{\{2r+s=2R+S\}}\left(2q-2r-s\right)!\left\langle f_n\widetilde{\otimes}_r^s f_n, f_n\widetilde{\otimes}_R^S f_n\right\rangle_{\mathfrak{H}^{\otimes(2q-2r-s)}} \\
&\quad + \sum_{r=1}^{q-1}\sum_{s=0}^{q-r} r\left(r!\right)^2\left(s!\right)^2\binom{q}{r}^4\binom{q-r}{s}^4\left(2q-2r-s\right)!\left\|f_n\widetilde{\otimes}_r^s f_n\right\|_{\mathfrak{H}^{\otimes(2q-2r-s)}}^2 + q\left(q!\left\|f_n\right\|_{\mathfrak{H}^{\otimes q}}^2\right)^2
\end{aligned}$$

We can again apply Hölder's inequality on the inner product, and conclude that all the terms go to 0 except the last term which goes to q. Therefore, $U_n \to q$ as $n \to \infty$.

Observe that

$$\begin{aligned}
\left|V_n\right| &\leq \mathbb{E}\left[\left\|x\left(DX_n\right)^2\right\|_{\mathfrak{H}}\left\|X_n\left(DX_n\right)\right\|_{\mathfrak{H}}\right] \leq \sqrt{\mathbb{E}\left[\left\|x\left(DX_n\right)^2\right\|_{\mathfrak{H}}^2\right]}\sqrt{\mathbb{E}\left[\left\|X_n\left(DX_n\right)\right\|_{\mathfrak{H}}^2\right]} \\
&= \sqrt{\mathbb{E}\left[\int x^2\left(D_z X_n\right)^4 d\mu\left(z\right)\right]}\sqrt{\mathbb{E}\left[\left\|X_n\left(DX_n\right)\right\|_{\mathfrak{H}}^2\right]}
\end{aligned}$$

Note that

$$\mathbb{E}\left[\left\|X_n\left(DX_n\right)\right\|_{\mathfrak{H}}^2\right] = \mathbb{E}\left[X_n^2\left\|DX_n\right\|_{\mathfrak{H}}^2\right] = U_n \to q.$$

From the third point, $\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}} x^2\left(D_z X_n\right)^4 d\mu\left(z\right)\right] \to 0$ so $V_n \to 0$ as $n \to \infty$.

Finally for W_n,

$$\left|W_n\right| = \mathbb{E}\left[\int \theta_z x^2\left(D_z X_n\right)^4 d\mu\left(z\right)\right] \leq \mathbb{E}\left[\int x^2\left(D_z X_n\right)^4 d\mu\left(z\right)\right] \longrightarrow 0.$$

Putting them together we get the fourth point: $\mathbb{E}[X_n^4] \to 3$ as $n \to \infty$. ■

In Wiener space, convergence in a fixed Wiener chaos to a standard normal distribution is characterized by the convergence of the fourth moments to 3 or of the convergence of the norm of certain contractions to 0 (see list preceding Corollary 21). We would then like to see if the same situation holds in Wiener-Poisson space. At this point, this appears to be an open question. We then finish with the following theorem which shows convergence in distribution and of the fourth moments to 3 if certain contractions converge to 0.

Theorem 29 *Let* $X_n = I_q(f_n)$*, with* $\mathbb{E}\left[X_n^2\right] = q!\,\|f_n\|^2_{\mathfrak{H}^{\otimes q}} \to 1$*. Assume that for* $r = 0, \ldots, q-1$ *and* $s = 0, \ldots, q-r$*,* $\|f_n \otimes_r^s f_n\|_{\mathfrak{H}^{\otimes(2q-2r-s)}} \mathbf{1}_{\{s=0,r\neq 0\}\cup\{s\neq 0,r=0\}} \to 0$*. Then as* $n \to \infty$*,*

- $\mathbb{E}[X_n^4] \to 3$.
- $X_n \to \mathcal{N}(0,1)$ *in distribution.*

Proof. The first assertion is the fourth point in Proposition 28. For the second point, we refer to Corollary 27 to see that it suffices to prove $g_*(X_n) - g_{X_n} \to 0$ in $L^2(\Omega)$ and $\left\langle |x|\,(DX_n)^2, |DX_n| \right\rangle_{\mathfrak{H}} \to 0$ in $L^1(\Omega)$ as $n \to \infty$. These are immediate when we note that

$$g_*(X_n) - g_{X_n} = 1 - \frac{1}{q}||DX_n||^2_{\mathfrak{H}} \to 0 \;\text{ in } L^2(\Omega) \qquad \text{(by point 2 of Proposition 28)}$$

and

$$\mathbb{E}\left[\left\langle |x|\,(DX_n)^2, |DX_n| \right\rangle_{\mathfrak{H}}\right] \leq \mathbb{E}\left[\left\|x\,(DX_n)^2\right\|_{\mathfrak{H}} \|DX_n\|_{\mathfrak{H}}\right] \leq \sqrt{\mathbb{E}\left[\left\|x\,(DX_n)^2\right\|^2_{\mathfrak{H}}\right]}\sqrt{\mathbb{E}\left[\|DX_n\|^2_{\mathfrak{H}}\right]}$$

$$= \sqrt{\mathbb{E}\left[\int x^2\,(DX_n)^4\,d\mu\right]}\sqrt{q\mathbb{E}\left[X_n^2\right]} \to 0 \qquad \text{(by point 3 of Proposition 28).}$$

■

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